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Courant Institute of
Mathematical Sciences

Magneto-Fluid Dynamics Division

The Stability of a Class of Bifurcated, Magnetohydrodynamic Free Boundary Equilibria

Jay Kappraff

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Introduction

One model for studying the dynamics of a contained plasma is called ideal magnetohydrodynamics. The plasma is assumed to be described by pressure, velocity, and density. The plasma is assumed to be in local thermodynamic equilibrium, and to be governed by the laws of conservation of mass, energy, and momentum. Lorentz forces, $\underset{\sim}{J} \times \underset{\sim}{B}$, are present but no electric forces are included. Forces arising from the electric field are ignored since density is assumed to be zero in a plasma. Finally, the plasma is assumed to be perfectly conducting, i.e., $\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B} = 0$, so that $\underset{\sim}{B}$ is determined by the Faraday law, $\underset{\sim}{B}_t = -\nabla \times \underset{\sim}{E} = \nabla \times (\underset{\sim}{u} \times \underset{\sim}{B})$. In this model, displacement current is ignored and $\underset{\sim}{J} = \nabla \times \underset{\sim}{B}$.

In this paper, we study the equilibrium and stability of a system consisting of a plasma column of length H, approximately a cylinder of radius r_o, surrounded by a vacuum region, all enclosed within an outer wall approximately a perfect conducting cylinder of radius, ρr_o. Vacuum magnetic fields $\underset{\sim}{B}^{in}$ and $\underset{\sim}{B}^{out}$ are present both in the plasma and vacuum regions, while surface currents are present on the plasma vacuum interface.

In equilibrium, the stress on the plasma vacuum interface from the plasma equals the stress from the external magnetic field. In this study, the plasma fluid pressure, p,

is constant, characterized by a dimensionless parameter, β, referred to as the plasma beta and defined as the ratio of the plasma fluid pressure to the total pressure both fluid and magnetic. Furthermore, we assume that the limiting values of $\underset{\sim}{B}^{in}$ and $\underset{\sim}{B}^{out}$ on the plasma vacuum interface are tangential to the surface, i.e. $\hat{n} \cdot \underset{\sim}{B} = 0$. Thus the condition for equilibrium is $[p + \frac{1}{2} B^2] = 0$, where [ ] denotes the magnitude of the discontinuity of a function across a surface. This equilibrium is called a free boundary equilibrium.

We study the stability of this equilibrium by following the analysis of Bernstein, et al. (Ref. 1), modified by Lust, et al. (Ref. 2). The dynamic equations of the plasma system are linearized by expressing the velocity, pressure, and magnetic field as the sum of their values at equilibrium, $\underset{\sim}{u}_o = 0$, $\underset{\sim}{B}_o$, p_o, and their perturbed values, or first variations, $\underset{\sim}{u}_1$, $\underset{\sim}{B}_1$, p_1, retaining only those terms linear in the perturbations. When the Lagrangian displacement, $\underset{\sim}{\xi}$, is introduced, where $\underset{\sim}{u} = \underset{\sim}{\xi}_t$, the linearized equations can be rewritten as a formally self-adjoint linear operator, L, mapping $\underset{\sim}{\xi}$ into its second derivative, i.e. $\underset{\sim}{\xi}_{tt} = L\underset{\sim}{\xi}$. If this equation is multiplied by $\underset{\sim}{\xi}_t$ and integrated over the entire system, the result is an expression for conservation of energy. The second variation of the potential energy, δW, is expressed here by the quadratic form $-(\underset{\sim}{\xi}, L\underset{\sim}{\xi})$.

The system is called stable if $\underset{\sim}{\xi}$ does not grow exponentially in time in an L_2 sense. If $\delta W \geq 0$ for all $\underset{\sim}{\xi}$, then the system is stable in this sense. The system is called unstable if there exist $\underset{\sim}{\xi}$ that grow exponentially in an L_2 sense, which is the case if there exist $\underset{\sim}{\xi}$ for which $\delta W < 0$.

In analogy with the theory of linear operators, we expect stationary states of δW to occur for eigenvalues of L. Thus, in keeping with the above definition, we will consider the system stable if all eigenvalues are non-negative, whereas the system is considered unstable if there exist negative eigenvalues.

In this study, we will show that for a certain class of equilibria characterized by ρ and other parameters, there exists a value of β referred to as beta critical, β_c, such that for all $\beta \leq \beta_c$, the system is stable, whereas for $\beta > \beta_c$ there exist unstable eigenstates. Corresponding to β_c, there exists a state, $\underset{\sim}{\xi}^{(o)}$, for which $\delta W = 0$. This state is referred to as marginally stable.

In this study, we will show that there exists a new family of equilibria in a neighborhood of $\underset{\sim}{\xi}^{(o)}$. This is an extension to plasma systems of the familiar concept of bifurcation encountered in certain systems governed by non-linear differential equations.

We will describe our equilibrium as a perturbation of system called a theta pinch. In a theta pinch, the magnetic field is directed along the axis of the column. In this study, we utilize predominantly theta pinch fields, to which we add a small admixture of fields which depend on the usual cylindrical coordinates ϕ and z only through the combination $\theta = \phi - kz$. Such fields are called helically symmetric, and k is called the helical wavenumber. These fields will be expanded as a Fourier series of terms of the form $r^m \exp [iL\theta]$ where L is an integer and $m \geq L - 1$ to insure regularity. For our study we consider fields consisting of a small quantity of L = 2 component whose ratio to the theta pinch field is characterized by a small parameter c, and an even smaller amount of L = 4 field, whose ratio to the theta pinch field is characterized by λc^2.

We assume long helical wavelengths, and introduce a second small dimensionless parameter, ε, defined as the plasma column radius times the helical wavenumber, i.e., $\varepsilon = kr_o$. We are then able to facilitate the mathematical analysis by expanding all geometrical and physical quantities as power series in ε and c, and carrying out a perturbation analysis of these expansions, as first ε and then c is made small. First, ε is made small enough to neglect all terms smaller than order ε^2. In the subsequent analysis, ε is considered fixed, leaving c as the remaining arbitrary, small

parameter. Then c is made small enough to retain only those leading order terms of significance to the analysis which follows.

Again, all terms past the leading order terms are assumed to be negligible. We emphasize that although we have not attempted to justify this assumption, our results depend strongly on its validity.

As mentioned above, the plasma pressure is characterized by the plasma beta. If the magnetic field in the plasma region were constant, since the fluid pressure is constant, the pressure would be effectively defined by specifying β. However, in our system, the magnetic field is only approximately constant, and correspondingly we define an approximate, constant β, which merely characterizes rather than defines the plasma pressure. For our study, β will be considered small, of order c^2, appropriate to an experimental device called a Stellarator.

The displacement, $\underset{\sim}{\xi}$, is expanded in a Fourier series of terms of the form $c_{m,n,\ell} r^m \exp [i(n\theta - \ell z)]$, where n and ℓ are the azimuthal and axial wavenumbers respectively, n takes on only integer values, $m \geq n - 1$ to insure regularity, and the coefficients $c_{m,n,\ell}$ are expressed as power series in ε and c, as mentioned above.

For any value of n, and for all small betas, we will show that there exists a family of unstable eigenstates.

These instabilities were also observed by Freidberg (Ref. 3) at large beta, and are called "kink instabilities." Since large values of n have not been observed in plasma systems, we will show that kink instabilities can be omitted by assuming $n \leq N$ and picking H correctly.

Once these instabilities are omitted, we will show that there exists a critical value of beta, as described above, and correspondingly a marginally stable state which serves as the starting point for bifurcation.

Bifurcation is a familiar concept encountered in certain systems governed by non-linear differential equations such as occurs in the buckling of beams. When the physical parameters of the system of a beam under compression reach a state called the bifurcation point, the equilibrium representing the straight beam becomes unstable to any further compression while another family of equilibria, representing the buckled state of the beam, becomes the stable configuration. As the beam is compressed further, it continuously branches to the more stable, buckled, configuration. It has been proven (Ref. 4), in the theory of differential equations, that for a conservative system, derivable from a Lagrangian, the bifurcation point is always a marginally stable state of the bifurcating equilibrium.

More recently, bifurcation analyses of hydromagnetic systems were studied, first by Friedrichs (Ref. 5), and then

by Yeh (Ref. 6). Friedrichs studied the stability of helical equilibria with small amplitude corrugations, to disturbances in the limit of high axial and azimuthal wavenumbers and found these equilibria unstable. Yeh also studied a helical system with small amplitude corrugations, but found stable bifurcated equilibria for n = 1 azimuthal wavenumbers.

In this study, we demonstrate the existence of bifurcated equilibria that are stable at states for which the original equilibria are unstable, provided the outer conducting wall is placed within 1.15 r_o of the plasma vacuum interface. We also show that the L = 4 fields do not affect the bifurcated stability.

Equilibrium

In this section we obtain a family of equilibria consisting of approximately L = 2 helically symmetric vacuum magnetic fields with small admixture of L = 4 fields for the outer region, a small plasma beta, with plasma vacuum interface slightly distorted from a cylinder. At the end of this section we append some geometrical preliminaries useful in the remainder of the thesis. This section and the next one provide no new results, but is basis for the bifurcation treatment to follow. The equilibrium will be expanded in terms of the small parameters, ε and c, introduced in the previous section.

We first formulate the helical equilibrium in a twisted coordinate system where $\hat{r}$, $\hat{\theta}$, and $\hat{\omega}$ are unit vectors in the directions $\underset{\sim}{\nabla} r$, $\underset{\sim}{\nabla}\theta = \underset{\sim}{\nabla}\phi - k\underset{\sim}{\nabla} z$, and $\hat{r} \times \hat{\theta}$, $\theta = \phi - kz$ being the helical angle introduced in the previous section. Thus $\hat{r}$, $\hat{\theta}$, and $\hat{\omega}$ are related to the cylindrical coordinates $\hat{r}$, $\hat{\phi}$, and $\hat{z}$ by

$$\hat{r} = \hat{r}$$

$$\hat{\theta} = \frac{1}{\sigma}(\hat{\phi} - kr\hat{z})$$

$$\hat{\omega} = \frac{1}{\sigma}(kr\hat{\phi} + \hat{z})$$

where $\sigma^2 = 1 + k^2r^2$.

Since, from Maxwell's equations, the fields are divergence free, we can express the helically symmetric vacuum field in terms of a flux function, $\psi(r,\theta)$, such that

$$\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\psi = 0 \qquad (1)$$

and

$$\underset{\sim}{B} = \underset{\sim}{\nabla}\psi \times \frac{\hat{\omega}}{\sigma} + B_\omega\hat{\omega} \quad , \qquad (2)$$

where in view of Maxwell's equation, $\underset{\sim}{J} \equiv \underset{\sim}{\nabla} \times \underset{\sim}{B} = 0$

$$B_\omega = \frac{\zeta}{\sigma}$$

for ζ constant, and ψ satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\sigma^2}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = \frac{2\zeta k}{\sigma^4} \quad . \tag{3}$$

Explicitly, (2) becomes

$$B_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$B_\theta = -\frac{1}{\sigma}\frac{\partial\psi}{\partial r}$$

$$B_\omega = \frac{\zeta}{\sigma}$$

For reasons described in the previous section, we wish to expand $\underset{\sim}{B}$ in the small parameter helical wavenumber, k. In order to proceed non-dimensionally, we introduce the small parameter, ε, defined as

$$\varepsilon = kr_o$$

where the characteristic plasma column radius, r_o, will be specified precisely later.

Following an idea of Weitzner (Ref. 7), we scale r and ψ by setting $r = \bar{r}r_o$ and $\psi = \varepsilon r_o\psi'$ and taking ζ to be independent of ε so that

$$B_r(\bar{r},\theta) = \frac{\varepsilon}{\bar{r}}\frac{\partial\psi'}{\partial\theta}$$

$$B_\theta(\bar{r},\theta) = -\varepsilon\frac{\partial\psi'}{\partial\bar{r}} + O(\varepsilon^3)$$

$$B_\omega = \zeta(1 - \varepsilon^2\frac{\bar{r}^2}{2}) + O(\varepsilon^4)$$

Equivalently, in cylindrical coordinates

$$B_r(\bar{r},\theta) = \frac{\varepsilon}{\bar{r}}\frac{\partial\psi'}{\partial\theta}$$

$$B_\phi(\bar{r},\theta) = \varepsilon(\zeta\bar{r} - \frac{\partial\psi'}{\partial\bar{r}}) + O(\varepsilon^3) \tag{4}$$

$$B_z(\bar{r},\theta) = \zeta + \varepsilon^2(\bar{r}\frac{\partial\psi'}{\partial\bar{r}} - \zeta\bar{r}^2) + (\varepsilon^4)$$

while (3) transforms to

$$\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left(\bar{r}\frac{\partial\psi'}{\partial\bar{r}}\right)+\frac{1}{\bar{r}^2}\frac{\partial^2\psi'}{\partial\theta^2}=2\zeta+O(\varepsilon^2)$$

We may express the above relations in two different ways appropriate for different purposes. First, if we work in the $\bar{r},\theta$ plane and define the usual operators $\bar{\nabla}$ and $\bar{\nabla}\cdot$ in that plane, and the vector $\bar{r}$ in that plane, then ψ' satisfies

$$\bar{\nabla}\cdot\bar{\nabla}\psi'=2\zeta+O(\varepsilon^2) \tag{5}$$

and from (4),

$$B^2=\zeta^2+\varepsilon^2(|\bar{\nabla}\psi'|^2-\zeta^2\,\bar{r}\cdot\bar{r})+O(\varepsilon^4) \tag{6}$$

Second, we may work in the $\bar{r},\phi,z$ space with the conventions that only the variables $\bar{r}$ and $\phi-kz$ appear and that $\bar{\nabla}$ and $\bar{\nabla}\cdot$ are the usual derivative operators in the $\bar{r}$ and ϕ plane only, and $\bar{r}$ is the position vector in this plane only, in which case (4) and (5) hold, and the magnetic field, expressed by (4), has the concise representation

$$B=\hat{z}[\zeta+\varepsilon^2(\bar{r}\cdot\nabla\psi-\zeta\bar{r}\cdot\bar{r})]+O(\varepsilon^4)-\varepsilon\hat{z}\times[\bar{\nabla}(\psi'-\tfrac{1}{2}\,\bar{r}\cdot\bar{r})+O(\varepsilon^2)] \tag{7}$$

In (7), we may extend $\bar{\nabla}$ to the derivative in the full $\bar{r},\phi,z$ space, provided $\bar{r}$ is still the position vector in the $\bar{r},\phi$ plane. While the various representations seem peculiar at present, they are particularly appropriate to facilitate the various transformations of coordinates which will be described in detail at the end of this section.

A free boundary problem is usually solved by giving the external field sources and determining the plasma-vacuum interface and fields. However, here we employ an inverse

procedure in which we specify a flux function for the plasma region satisfying (5). The flux function for the vacuum region is then determined, satisfying (5) and the boundary conditions for free boundary equilibria, namely, that the flux function be constant at the plasma vacuum interface as required by (1), and be chosen so as to achieve a pressure balance across the interface.

This equation and boundary conditions constitutes a Cauchy problem for which the Cauchy-Kowalewski theorem guarantees a solution for ψ, the plasma vacuum interface, and the outer flux surface in some neighborhood of the plasma vacuum interface as ε approaches zero for small but finite c. In general, a singularity in the solution appears within the vacuum region for finite c. However, as c approaches zero, the singularity moves beyond the outer flux surface. We are then able to justify our expansion techniques by employing a known existence theorem for free boundary equilibria (Ref. 8). Once we have solved the inverse problem, this theorem can be used to solve the direct problem in which the outer flux surface is taken as the initial surface for the exact solution of the free boundary equilibrium and the plasma vacuum interface determined. It is expected that this interface will be close to the one computed for the inverse problem.

We proceed to construct the equilibrium using this inverse procedure.

We take the flux function for the plasma region to be

$$\psi^{in} = \gamma\left(\frac{\bar{r}^2}{2} - \varepsilon c \frac{\bar{r}^2}{2} \cos 2\theta + \lambda \varepsilon c^2 \bar{r}^4 \cos 4\theta\right) \tag{8}$$

which is a solution of (5) with the constant γ replacing ζ.

For the subsequent analysis, ψ will be written without primes.

From (7), we determine the field in the plasma:

$$\begin{aligned} \underset{\sim}{B}^{in} = \gamma[&\hat{z} + \varepsilon c \bar{r}(\sin 2\theta \hat{r} + \cos 2\theta \hat{\phi}) \\ &- 4\lambda \varepsilon c^2 \bar{r}^3(\sin 4\theta \hat{r} + \cos 4\theta \hat{\phi})] \end{aligned} \tag{9}$$

The radius of the plasma-vacuum interface is expanded in a Fourier series:

$$\bar{r} = 1 + \alpha_2 \cos 2\theta + \alpha_4 \cos 4\theta + \alpha_6 \cos 6\theta + \ldots \tag{10}$$

From the condition that ψ^{in} be constant at the plasma surface, we solve for α_2, α_4, and α_6 order by order in c. The result is

$$\begin{aligned} \alpha_2 &= \frac{c}{2} + \left(\frac{9}{64} - \frac{5}{4}\lambda\right)c^3 \\ \alpha_4 &= \left(\frac{3}{16} - \lambda\right)c^2 \\ \alpha_6 &= \left(\frac{1}{8} - \frac{3}{2}\lambda\right)c^3 \end{aligned} \tag{11}$$

For the vacuum region, since ψ^{out} must also satisfy (5), we can with all generality, set

$$\begin{aligned} \psi^{out} = \zeta\Big(&\frac{\bar{r}^2}{2} + \varepsilon c_\infty \log \bar{r} - c_2 \varepsilon c \bar{r}^2 \cos 2\theta + c_{-2} \varepsilon c \bar{r}^{-2} \cos 2\theta \\ &+ c_4 \varepsilon c^2 \bar{r}^4 \cos 4\theta + c_{-4} \varepsilon c^2 \bar{r}^{-4} \cos 4\theta + \ldots\Big) \end{aligned} \tag{12}$$

We take $c_\infty = 0$ so that the net current in the free boundary is zero. The inclusion of c_∞ is possible, but experience suggests that it leads to less stable equilibria.

Since the plasma pressure supported by the field, at equilibrium, is just $[\frac{B^2}{2}]$, the plasma beta is

$$\beta = \frac{P}{p + \frac{B^2 \text{ in}}{2}} = \frac{[B^2]}{B^2 \text{ out}} \quad ,$$

and since, from (6),

$$[B^2] = \zeta^2 - \gamma^2 + O(\epsilon^2),$$

$$\beta = (1 - (\frac{\gamma}{\zeta})^2)\ (1 + O(\epsilon^2)) \ .$$

Without serious error, we define

$$\beta = 1 - (\frac{\gamma}{\zeta})^2 \quad . \tag{13}$$

The remaining boundary conditions for free boundary equilibria are,

$$\psi^{\text{out}}(\bar{r}) = \text{constant}$$

and,

$$[B^2] = \text{constant}$$

at the plasma surface.

These conditions, along with (6), (10), (11), (12), and (13) and the assumption that plasma beta, β, is small of order c^2, enable us to solve order by order in c for the coefficients c_2, c_{-2}, c_4, and c_{-4} of (12). After a long computation

$$\begin{aligned} c_2 &= \frac{c}{2} - \frac{c\beta}{4} \\ c_{-2} &= -\frac{c\beta}{4} \\ c_4 &= \lambda c^2 + O(c^4) \\ c_{-4} &= O(c^4) \end{aligned} \tag{14}$$

Replacing these coefficients in (7), we have for the outer vacuum fields,

$$\underset{\sim}{B}^{out} = \zeta\Big[\big(\hat{z}+c\varepsilon\bar{r}(\sin 2\theta\hat{r}+\cos 2\theta\hat{\phi})-\frac{c\beta}{2}\,\varepsilon\bar{r}(\sin 2\theta r+\cos 2\theta\hat{\phi})$$

$$+\,\varepsilon\frac{c\beta}{2}\,\bar{r}^{-3}(\sin 2\theta\hat{r}-\cos 2\theta\hat{\phi})-4\lambda c^2\,\bar{r}^{3}(\sin 4\theta\hat{r}+\cos 4\theta\hat{\phi})\Big]$$

$$+\ (\text{higher order terms in } \varepsilon \text{ and } c) \quad . \qquad (15)$$

Thus, the inverse procedure has yielded the desired vacuum magnetic fields.

In addition, we see from (10) that the equilibrium configuration of the plasma surface is that of a cylinder with small helical corrugations of width, order c, with respect to the characteristic radius, r_o, of the cylinder.

We conclude this section by giving some useful geometrical preliminaries, associated with the several coordinate transformations introduced earlier in this section.

We define rectangular coordinates corresponding to the $\bar{r}$, θ coordinates defined in this section, as follows:

$$x' = \bar{r}\cos\theta = \frac{x}{r_o}\cos kz + \frac{y}{r_o}\sin kz$$

$$y' = \bar{r}\sin\theta = \frac{-x}{r_o}\sin kz + \frac{y}{r_o}\cos kz$$

and extend these coordinates from the x', y'-plane to a transformation of x,y,z-space to x',y',z'-space as follows:

$$r_o x' = x\cos kz + y\sin kz$$

$$r_o y' = -x\sin kz + y\cos kz \qquad (16)$$

$$r_o z' = \varepsilon z$$

These relations emphasize that, by studying long helical wavelengths, we are assuming slow z dependence relative to x or y dependence. If we define the corresponding system of orthogonal vectors $\hat{x}'$, $\hat{y}'$, $\hat{z}'$ as

$$\begin{aligned} \hat{x}' &= \hat{x} \cos kz + \hat{y} \sin kz \\ \hat{y}' &= -\hat{x} \sin kz + \hat{y} \cos kz \\ \hat{z}' &= \hat{z} \end{aligned} \tag{17}$$

the $\hat{x}'$ and $\hat{y}'$ vectors are not independent of z', in fact

$$\frac{\partial \hat{x}'}{\partial z'} = \hat{y}' \qquad \text{and} \qquad \frac{\partial \hat{y}'}{\partial z'} = -\hat{x}'$$

so that when we introduce the standard polar coordinates, r', θ', in the x', y' plane, previously referred to as $\bar{r}$, θ, and the unit polar vectors $\hat{r}'$, $\hat{\theta}'$, we see that

$$\frac{\partial \hat{r}'}{\partial r'} = \frac{\partial \hat{\theta}'}{\partial r'} = 0$$

$$\frac{\partial \hat{r}'}{\partial z'} = \hat{\theta}' = \frac{\partial \hat{r}'}{\partial \theta'}$$

and

$$\frac{\partial \hat{\theta}'}{\partial z'} = -\hat{r}' = \frac{\partial \hat{\theta}'}{\partial \theta'}$$

With the above vector identities, we may rewrite any vector relation given in terms of x,y,z or r,ϕ,z in terms of x',y',z' or r',θ',z'. For example, it is easy to show that

$$\underset{\sim}{\nabla} = \frac{1}{r_o}\left[\hat{r}' \frac{\partial}{\partial r'} + \frac{\hat{\theta}'}{r'} \frac{\partial}{\partial \theta'} + \varepsilon \hat{z}' \left(\frac{\partial}{\partial z'} - \frac{\partial}{\partial \theta'}\right)\right] \tag{18}$$

and,

$$\nabla \times \underset{\sim}{T} = \left(\frac{1}{r'}\frac{\partial T^{z'}}{\partial \theta'} - \varepsilon\left(\frac{\partial}{\partial \theta'}, - \frac{\partial}{\partial \theta'}, T^{\theta'}\right)\hat{r}' + \left(-\frac{\partial (T^{z'})}{\partial r'} + \varepsilon\left(\frac{\partial}{\partial z'}, - \frac{\partial}{\partial \theta'}\right)T^{r'}\right)\hat{\theta}'\right.$$

$$\left. + \left(\frac{1}{r'}\left(\frac{\partial (r'T)}{\partial r'} - \frac{\partial T^{r'}}{\partial \theta'}\right)\right)\hat{z}' \right. \tag{19}$$

If we specify a surface in r',θ',z'-space by $r' = r'(\theta',z')$, then the parametric representation of this surface is

$$\underset{\sim}{r}(\phi,z) = r(\phi,z)\hat{r} + z\hat{z}$$

or,

$$\underset{\sim}{r}(\theta',z') = r_o r'(\theta')\hat{r}' + r_o \frac{z'}{\varepsilon}\hat{z}'$$

Two tangent vectors to this surface are $\frac{\partial \underset{\sim}{r}}{\partial z'}$ and $\frac{\partial \underset{\sim}{r}}{\partial \theta'}$, so that a normal vector to the surface is:

$$\underset{\sim}{\bar{n}} = \frac{1}{r' r_o}\frac{\partial \underset{\sim}{r}}{\partial \theta'} \times \frac{\varepsilon}{r_o}\frac{\partial \underset{\sim}{r}}{\partial z'} = \hat{r}' - \frac{\hat{\theta}'}{r'}\frac{\partial r'}{\partial \theta'} - \varepsilon\frac{\partial r'}{\partial z'}\hat{z}' \tag{20}$$

The element of surface area is:

$$dS = \left|\left(\frac{\partial \underset{\sim}{r}}{\partial \theta'} \times \frac{\partial \underset{\sim}{r}}{\partial z'}\right)\right| d\theta' dz'$$

Thus, at the plasma–vacuum interface:

$$dS = \frac{r_o}{k}\, d\theta' dz' r' \left|\underset{\sim}{\bar{n}}\right| \left(1 + O(\varepsilon^2)\right) \tag{21}$$

It is easy to show that although the set of unit vectors $\hat{x}',\hat{y}',\hat{z}'$ differ from $\hat{x},\hat{y},\hat{z}$, the corresponding sets of polar vectors $\hat{r}',\hat{\theta}',\hat{z}'$ and $\hat{r},\hat{\phi},\hat{z}$ are identical. Thus, any vector equation written in terms of $\hat{r},\hat{\phi},\hat{z}$, can be re-written identically in terms of $\hat{r}',\hat{\theta}',\hat{z}'$, and for the remainder of this paper, the primes will be omitted, and all vectors will be referred to the $\hat{r},\hat{\theta},\hat{z}$ system and the coordinates referred to, without primes, as r,θ,z. Thus,

equation (9) can be rewritten as:

$$\underset{\sim}{B}^{in} = \gamma\hat{z} + \varepsilon cr(\sin 2\theta\hat{r} + \cos 2\theta\hat{\theta}) - 4\lambda\varepsilon c^2 r^3(\sin 4\theta\hat{r} + \cos 4\theta\hat{\theta}) \tag{9'}$$

Also (10), (20), and (21) can be rewritten, dropping terms of order, $O(c^3)$ and $O(\varepsilon^2)$, as:

$$r = 1 + \frac{c}{2}\cos 2\theta + (\frac{3}{16} - \lambda)c^2\cos 4\theta \tag{10'}$$

$$\underset{\sim}{\bar{n}} = \hat{r} + (c \sin 2\theta + c^2(\frac{1}{2} - 4\lambda) \sin 4\theta\hat{\theta}) + O(\varepsilon)\hat{z} \tag{20'}$$

and,

$$dS = \frac{r_o}{k} |\underset{\sim}{\bar{n}}| \, rd\theta\phi z \tag{21'}$$

A final useful preliminary computation that follows from (6), (8), (12), (14), and (20) is:

$$\hat{n}\cdot\nabla\frac{[\underset{\sim}{B}^2]}{2} = \zeta^2\varepsilon^2[\frac{c^2\beta}{2}(r-r^{-3}) - c\beta(r+r^{-3})\cos 2\theta] \tag{22}$$

where

$$[\underset{\sim}{B}^2] = \zeta\varepsilon c\beta(-r^2 + r^{-2})\cos 2\theta \quad . \tag{22'}$$

Stability

We now commence with the stability analysis described in the Introduction, in which we study the second variation of the potential energy. The second variation of the potential energy, δW, consists of three terms, a volume integral, δW^P, over the plasma region; a surface integral, δW^Σ, over the plasma–vacuum interface; and another volume integral, δW^V, over the vacuum region. The system is considered stable if the sum of the three contributions is positive for all admissible plasma displacements, $\underset{\sim}{\xi}(r,\theta,z)$, and disturbed scalar potentials of the vacuum fields, $\Phi(r,\theta,z)$, where Φ and $\underset{\sim}{\xi}$ are related at the interface, and $\frac{\partial \Phi}{\partial n} = 0$ on the bounding outer surface, assumed to be a perfect conductor.

For this study, as we mentioned above, we are interested in determining a critical beta and a corresponding, marginally stable state to serve as the starting point for bifurcation. Thus, we need only determine the lowest eigenvalues and associated eigenfunctions of δW. Therefore, it is sufficient to determine $\underset{\sim}{\xi}$ and Φ so as to minimize the two volume integrals, holding $\hat{n} \cdot \underset{\sim}{\xi}$ fixed on the plasma vacuum interface. As a result, $\underset{\sim}{\xi}$ and Φ satisfy Euler variational equations, which, in effect, reduce the volume integrals to surface integrals over the plasma vacuum interface.

More precisely,

$$\delta W = \delta W^P + \delta W^\Sigma + \delta W^V$$

where,

$$2\delta W^{p} = \int dv \{ [\Gamma p(\nabla_{\sim}\cdot\xi_{\sim})^2] + [\nabla_{\sim}\times(\xi_{\sim}\times B_{\sim})]^2\} \quad , \qquad (23)$$

$$2\delta W^{\Sigma} = \int dS(\hat{n}\cdot\xi_{\sim})^2\, \hat{n}\cdot\nabla_{\sim}\frac{[B^2]}{2} \quad , \qquad (24)$$

and

$$2\delta W^{V} = \int dv(\nabla_{\sim}\Phi)^2 \quad , \qquad (25)$$

while on the interface:

$$\hat{n}\cdot\nabla_{\sim}\Phi = \hat{n}\cdot\nabla_{\sim}\times(\xi_{\sim}\times B_{\sim}^{out}) \qquad (26)$$

where Γ in equation (23) is the ratio of specific heats.

Provided that the Euler equations

$$0 = \Gamma p\nabla_{\sim}(\nabla_{\sim}\cdot\xi_{\sim}) - B_{\sim}\times\{\nabla_{\sim}\times[\nabla_{\sim}\times(\xi_{\sim}\times B_{\sim})]\} \quad , \qquad (27)$$

$$0 = \nabla_{\sim}\cdot\nabla_{\sim}\Phi \qquad (28)$$

are satisfied, we may rewrite equations (23) and (25) as

$$2\delta W^{p} = \int dS(\hat{n}\cdot\xi_{\sim})[\Gamma p\nabla_{\sim}\cdot\xi_{\sim} - B_{\sim}\cdot\nabla_{\sim}\times(\xi_{\sim}\times B_{\sim})] \qquad (23')$$

$$2\delta W^{V} = \int dS\Phi\hat{n}\cdot\nabla_{\sim}\Phi \qquad (25')$$

The only negative term in δW comes from equation (24) [c.f. (23) – (25)]. In view of (22), we see that if $\xi_{\sim}$ (and $\hat{n}\cdot\xi_{\sim}$) are O(1) in ε, then unless all positive terms in equations (23) and (25) are of order ε^2 or smaller, the system is automatically stable. In particular, we infer that

$$\nabla_{\sim}\cdot\xi_{\sim} = O(\varepsilon) \qquad (29)$$

and

$$\nabla_{\sim}\times(\xi_{\sim}\times B_{\sim}) = O(\varepsilon) \qquad (30)$$

If we expand $\underset{\sim}{\xi}$ in a formal series in ε

$$\underset{\sim}{\xi}(r,\theta,z) = \underset{\sim}{\xi}_0(r,\theta,z) + \varepsilon\underset{\sim}{\xi}_1(r,\theta,z) + \varepsilon^2\underset{\sim}{\xi}_2(r,\theta,z) + \ldots \quad (31)$$

then, from equations (18), (30), and (32):

$$\frac{1}{r}\left\{ \frac{\partial(r\xi_0^r(r,\theta,z))}{\partial r} + \frac{\partial \xi_0^\theta}{\partial \theta} \right\} = O(\varepsilon) \quad . \quad (32)$$

We may now rearrange terms among $\underset{\sim}{\xi}_0$ and $\underset{\sim}{\xi}_1$ in equation (32), leaving $\underset{\sim}{\xi}$ fixed so that

$$\frac{1}{r}\left\{ \frac{\partial(r\xi_0^r)}{\partial r} + \frac{\partial \xi_0^\theta}{\partial \theta} \right\} = 0 \quad (32')$$

or the divergence of $\underset{\sim}{\xi}$ restricted to the r,θ plane vanishes. Equation (32') is a more convenient form of equation (32).

We now start the evaluation of δW^p. We may rewrite the Euler variational equation, (27), as

$$\underset{\sim}{\nabla}\Omega = \nabla[p\Gamma(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi}) - \underset{\sim}{B}\cdot\underset{\sim}{Q}] = -\underset{\sim}{Q}\cdot\nabla\underset{\sim}{B} - \underset{\sim}{B}\cdot\nabla\underset{\sim}{Q} \quad (33)$$

where, in view of equation (32')

$$\underset{\sim}{Q} = \nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}) = (\underset{\sim}{B}\cdot\nabla)\underset{\sim}{\xi} - (\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla})\underset{\sim}{B} + O(\varepsilon^2) \quad . \quad (34)$$

We set

$$\underset{\sim}{B} = \underset{\sim}{B}_0 + \varepsilon\underset{\sim}{B}_1$$

and find that

$$\underset{\sim}{Q} = \underset{\sim}{\nabla}\times[(\underset{\sim}{\xi}_0\times\underset{\sim}{B}_0) + (\underset{\sim}{\xi}_0\times\varepsilon\underset{\sim}{B}_1) + (\varepsilon\underset{\sim}{\xi}_1\times\underset{\sim}{B}_0) + O(\varepsilon^2)] \quad .$$

In view of equations (18) and (32'), to order ε,

$$\underset{\sim}{Q} = \gamma\varepsilon\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial\theta}\right)\underset{\sim}{\xi}_0 + \varepsilon(B_1\cdot\underset{\sim}{\nabla})\underset{\sim}{\xi}_0 - (\underset{\sim}{\xi}_0\cdot\underset{\sim}{\nabla})\varepsilon\underset{\sim}{B}_1 - \gamma\varepsilon(\underset{\sim}{\nabla}\cdot\underset{\sim}{\xi}_1)\hat{z}$$

Denoting

$$\underset{\sim}{Q}_o = \gamma\varepsilon\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial\theta}\right)\underset{\sim}{\xi}_o$$

and

$$\underset{\sim}{Q}_1 = \varepsilon(\underset{\sim}{B}_1\cdot\underset{\sim}{\nabla})\underset{\sim}{\xi}_o - \varepsilon(\underset{\sim}{\xi}_o\cdot\underset{\sim}{\nabla})\underset{\sim}{B}_1$$

then,

$$-\underset{\sim}{Q}\cdot\underset{\sim}{\nabla}\underset{\sim}{B} - \underset{\sim}{B}\cdot\underset{\sim}{\nabla}\underset{\sim}{Q} = -\varepsilon(\underset{\sim}{Q}_o+\underset{\sim}{Q}_1)\cdot\underset{\sim}{\nabla}\underset{\sim}{B}_1 - (\underset{\sim}{B}_o+\varepsilon\underset{\sim}{B}_1)\cdot\nabla(\underset{\sim}{Q}_o+\underset{\sim}{Q}_1)$$

$$+ O(\varepsilon^3) + O(\varepsilon^2)\hat{z} \qquad (35)$$

Since the latter expression is the gradient of a scalar function, its curl must vanish. And, since the leading order term in this expression is order ε^2 from equation (19'), the vanishing of the $\hat{r}$ and $\hat{\theta}$ components of the curl imply that the coefficient of the $\hat{z}$ is of order ε^3 rather than ε^2. Thus, if we are interested only in the leading order terms in ε, all terms of order ε^3 are negligible. Since the leading order terms of equation (35) involve only the $\underset{\sim}{\xi}_o$ component of $\underset{\sim}{\xi}$, we fix ε at a value, small enough to neglect terms of order ε^3, and with no ambiguity, refer to $\underset{\sim}{\xi}_o$ as $\underset{\sim}{\xi}$ for the remainder of this paper.

We proceed to expand $\underset{\sim}{\xi}$ in terms of the other small parameter, c. Since $-\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\underset{\sim}{Q} - \underset{\sim}{Q}\cdot\underset{\sim}{\nabla}\underset{\sim}{B}$ is the gradient of a potential Ω, its curl vanishes, and from the z-component of the curl, we solve order by order in c for the r,θ components ξ^r and ξ^θ of $\underset{\sim}{\xi}$. These components are O(1) in ε. Again, from the vanishing of the $\hat{r},\hat{\theta}$ components of $\underset{\sim}{\nabla}\times[-\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\underset{\sim}{Q} - \underset{\sim}{Q}\cdot\underset{\sim}{\nabla}\underset{\sim}{B}]$, ξ^z is found to be $O(\varepsilon)$ and hence contributes only terms of $O(\varepsilon^3)$ to $-\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\underset{\sim}{Q} - \underset{\sim}{Q}\cdot\underset{\sim}{\nabla}\underset{\sim}{B}$. Thus, ξ^z does

not affect this analysis.

More precisely, we set

$$\underset{\sim}{\xi} = \underset{\sim}{\xi}_o + c\underset{\sim}{\xi}_1 + c^2\underset{\sim}{\xi}_2 + \ldots$$

$$\underset{\sim}{B} = \underset{\sim}{B}_o + c\underset{\sim}{B}_1 + c^2\underset{\sim}{B}_2 + \ldots \qquad (36)$$

$$\underset{\sim}{Q} = \underset{\sim}{Q}_o + c^2\underset{\sim}{Q}_1 + c^2\underset{\sim}{Q}_2 + \ldots$$

where the subscripts now refer to the order of c appearing explicitly in the expression for $\underset{\sim}{\xi}$, $\underset{\sim}{B}$, and $\underset{\sim}{Q}$.

In (36), $\underset{\sim}{B}_i$, i = 0,1,2 is given by (19'), while from the definition of $\underset{\sim}{Q}$,

$$\underset{\sim}{Q}_j = \sum_{i=o}^{j} [(\underset{\sim}{B}_i \cdot \nabla)\underset{\sim}{\xi}_{j-i} - (\underset{\sim}{\xi}_{j-i} \cdot \nabla)\underset{\sim}{B}_i] \qquad (37)$$

where $\underset{\sim}{B}_i = 0$ for i > 2. From the Euler equation (33)

$$\{\nabla \times \{-\underset{\sim}{B} \cdot \nabla \underset{\sim}{Q} - \underset{\sim}{Q} \cdot \nabla \underset{\sim}{B}\}\}_z = 0 \qquad (38)$$

Furthermore, since $\underset{\sim}{\xi}$ is divergence free, i.e.,

$$\nabla \cdot \underset{\sim}{\xi} = \frac{1}{r}\left(\frac{\partial (r\xi^r)}{\partial r} + \frac{\partial \xi^\theta}{\partial \theta}\right) = 0 \quad , \qquad (32'')$$

we can introduce the flux function, χ, where

$$\xi_i^r = \frac{1}{r}\frac{\partial \chi_i}{\partial \theta} \quad ,$$

$$\xi_i^\theta = -\frac{\partial \chi_i}{\partial r} \quad , \quad \text{and}$$

$$\chi = \chi_o + c\chi_1 + c^2\chi_2 + \ldots$$

If we define the linear operator $\mathcal{O}$ by

$$\mathcal{O} = \gamma\varepsilon\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \theta}\right)$$

then

$$\mathcal{O}\,\underset{\sim}{\xi}_i \equiv (\underset{\sim}{B}_o\cdot\underset{\sim}{\nabla})\underset{\sim}{\xi}_i$$

and the Euler equation (41) can be rewritten order by order in c as

$$\mathcal{O}^2\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\chi_o = 0 \qquad (39a)$$

$$\mathcal{O}^2\underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\chi_j = \sum_{i=1}^{j}\{\underset{\sim}{\nabla}\times[\underset{\sim}{B}_i\cdot\nabla\underset{\sim}{Q}_{j-i} + \underset{\sim}{Q}_{j-i}\cdot\nabla\underset{\sim}{B}_i]\}_z \quad , \quad j \geq 1 \qquad (39b)$$

We now proceed to evaluate $\underset{\sim}{\xi}$. Since the equilibrium, to any order in c, is a function of r and θ alone, the Euler equations do not contain z explicitly, and we may thus assume that both $\underset{\sim}{\xi}$ and Φ have z-dependence, $\exp(-i\ell z)$. Unless $\underset{\sim}{\xi}_o$ has θ and z dependence, $\exp[-i\ell(\theta+z)]$, we see from (39a) that $\underset{\sim}{\xi}_o$ is an harmonic vector. The special case of θ and z dependence, $\exp[-i\ell(\theta+z)]$, will be considered separately. It should be pointed out, however, that we can take $\ell \neq 0$ in this expression, for if $\ell = 0$, regularity of ξ_o^r at $r = 0$, and (32") imply that $\xi_o^r \equiv 0$. Thus we may assume, with all generality that (ξ_o^r, ξ_o^θ) is harmonic in the plane. Thus

$$\underset{\sim}{\xi}_o = \sum_{n=1}^{\infty} n\delta_{n,\ell}\, r^{n-1}\exp[i(n\theta-\ell z)](i\hat{r} - \hat{\theta})$$

$$+ \sum_{n=-1}^{-\infty} n\delta_{n,\ell}\, r^{-n-1}\exp[i(n\theta - \ell z)](i\hat{r} + \hat{\theta}) \qquad (40)$$

Replacing $\underset{\sim}{\xi}_o$ in (39b), and noting that $\underset{\sim}{\xi}_o$ for $n < 0$, ℓ modes are merely complex conjugates of corresponding $n > 0$, ℓ modes, we have

$$\mathcal{O}^2 \underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\chi_1 = \sum_{n=1}^{\infty} -4\gamma^2\varepsilon^2 n(n-1)(\ell+n-2)\delta_{n,\ell} r^{n-2}\exp[i\{(n-2)\theta-\ell z\}] \quad (41)$$

Assuming,

$$\chi_1 = -\frac{\alpha_{n,n-2}\, r^n \exp[i\{(n-2)\theta - \ell z\}]}{(\ell + n - 2)^2}$$

and equating coefficients of like terms in (41), we solve for χ_1 and hence $\underset{\sim}{\xi}_1$. The result is

$$\underset{\sim}{\xi}_1 = \sum_{n>1} n\delta_{n,\ell}\, r^{n-1} \frac{\exp[i\{(n-2)\theta - \ell z\}]\{(n-2)i\hat{r} - n\hat{\theta}\}}{\ell + n - 2}$$

$$+ \text{ (complex conjugate modes)} \quad (42)$$

for $n \neq \pm 1$ and $\ell + n \pm 2 \neq 0$. The arbitrary coefficients $\delta_{n,\ell}$ will be determined subsequently so as to render the form δW stationary. Conditions which render the Euler equation, (39b), homogeneous will be referred to as singular conditions, and the corresponding harmonic solutions will be called singular solutions. Thus, from equation (41), it follows that $n = \pm 1$ or $\ell + n \pm 2 = 0$ are singular conditions and the corresponding solutions are harmonic.

In a similar way,

$$\mathcal{O}^2 \underset{\sim}{\nabla}\cdot\underset{\sim}{\nabla}\chi_2 = \sum_{n=1}^{\infty} 48\lambda\gamma^2\varepsilon^2 n(n-1)(\ell+n-4)\delta_{n,\ell} r^n \exp[i\{(n-4)\theta-\ell z\}]$$

$$\sum_{n=1}^{\infty} -8\gamma^2\varepsilon^2 n(n-1)(n-2)(\ell+n-4)\delta_{n,\ell} \frac{r^{n-2}\exp[i\{(n-4)\theta-\ell z\}]}{\ell + n - 2}$$

from which it follows that for $\ell + n \pm 4 \neq 0$, $\ell + n \pm 2 \neq 0$

$$\underset{\sim}{\xi}_2 = \sum_{n>2} \gamma^2\varepsilon^2 n(n-1)\delta_{n,\ell} \frac{r^{n-1}\exp[i\{(n-4)\theta-\ell z\}]\{(n-4)i\hat{r}-n\hat{\theta}\}}{(\ell+n-4)(\ell+n-2)}$$

$$\sum_{n>2} -4\lambda\gamma^2\varepsilon^2 n\delta_{n,\ell} \frac{r^{n+1}\exp[i\{(n-4)\theta-\ell z\}]\{(n-4)i\hat{r}-(n+2)\hat{\theta}\}}{\ell + n - 4}$$

$$+ \text{ (complex conjugate modes)} \quad (43)$$

Again the singular solutions, $n = \pm 1$, ± 2 and $\ell + n \pm 4 = 0$ are harmonic.

From (19'), (40), (42) and (43), and in view of (37), the non-singular solutions yield

$$\underset{\sim}{Q}_o = \sum_{n>o} -i\gamma\varepsilon n(\ell + n)\delta_{n,\ell}\, r^{n-1} \exp[i(n\theta - \ell z)](i\hat{r} - \hat{\theta})$$

$$+ \text{(complex conjugate modes)}$$

$$\underset{\sim}{Q}_1 = 0$$

$$\underset{\sim}{Q}_2 = \sum_{n>2} -i\gamma\varepsilon n^2 \delta_{n,\ell} \frac{r^{n-1} \exp[i(n\theta - \ell z)](i\hat{r} - \hat{\theta})}{(\ell + n - 2)} \qquad (44)$$

$$+ \sum_{n>2} i\gamma\varepsilon n(n-1)\delta_{n,\ell}\, \frac{r^{n-1} \exp[i\{(n-4)\theta - \ell z\}]\{(n-4)\theta i\hat{r} - n\hat{\theta}\}}{\ell + n - 2}$$

$$+ \sum -4\lambda i\gamma\varepsilon n \delta_{n,\ell} r^{n+1} \exp[i\{(n-4)\theta - \ell z\}]\{(n-4)i\hat{r} - (n+2)\hat{\theta}\}$$

$$+ \text{(complex conjugate modes)}$$

Next, Ω is expanded in a formal series:

$$\Omega = \Omega_o + c\Omega_1 + c^2\Omega_2 + \ldots$$

and computed from its defining equation, (33). For the non-singular solutions, the result is

$$\Omega_o = \sum_{n\geq 1} i\gamma^2\varepsilon^2(\ell + n)^2\delta_{n,\ell}\, r^n \exp[i(n\theta - \ell z)]$$

$$+ \text{(complex conjugate)}$$

$$\Omega_1 = \sum_{n>1} -i\gamma^2\varepsilon^2 n(\ell + n)\delta_{n,\ell}\, r^n \exp[i\{(n-2)\theta - \ell z\}] \qquad (45)$$

$$+ \text{(complex conjugate)}$$

$$\Omega_2 = \sum_{n>1} 4\lambda i\gamma^2\varepsilon^2 n(\ell + n)\delta_{n,\ell} \frac{r^{n+2}}{\ell + n - 2} \exp[i((n-4)\theta - \ell z)$$

$$+ \text{(complex conjugate)} \quad .$$

For the singular $n = \pm 1$ solutions

$$\underset{\sim}{Q}_o = - i\gamma\varepsilon(\ell + 1)\delta_{1,\ell} \exp[i(\theta - \ell z)](i\hat{r} - \hat{\theta})$$

$$+ i\gamma\varepsilon(\ell - 1)\delta_{-1,\ell} \exp[i(-\theta - \ell z)(i\hat{r} + \hat{\theta})$$

$$\underset{\sim}{Q}_1 = - i\gamma\varepsilon[\delta_{1,\ell} \exp[i(-\theta - \ell z)](i\hat{r} + \hat{\theta}) \qquad (46)$$

$$+ \delta_{-1,\ell} \exp[i(\theta - \ell z)](i\hat{r} - \hat{\theta})]$$

$$\underset{\sim}{Q}_2 = 12\lambda i\gamma^2\varepsilon^2\delta_{-1,\ell} r^2 \exp[i(3\theta - \ell z)](i\hat{r} - \hat{\theta})$$

$$+12\lambda i\gamma^2\varepsilon^2\delta_{1,\ell} r^2 \exp[i(-3\theta - \ell z)](i\hat{r} + \hat{\theta})$$

whereas, for the singular $n = \pm 1$ solutions

$$\Omega_o = i\gamma^2\varepsilon^2(\ell + 1)^2\delta_{1,\ell} r \exp[i(\theta - \ell z)]$$

$$- i\gamma^2\varepsilon^2(\ell - 1)^2\delta_{-1,\ell} r \exp[i(-\theta - \ell z)]$$

$$\Omega_1 = - 2i\gamma^2\varepsilon^2\delta_{1,\ell} r \exp[i(-\theta - \ell z)]$$

$$+ 2i\gamma^2\varepsilon^2\delta_{-1,\ell} r \exp[i(\theta - \ell z)]$$

$$\Omega_2 = -4\lambda i(\ell-3)\gamma^2\varepsilon^2\delta_{1,-1} r^3 \exp[i(-3\theta - \ell z)] \qquad (47)$$

$$-4i\lambda(\ell+3)\gamma^2\varepsilon^2\delta_{-1,-1} r^3 \exp[i(3\theta - \ell z)]$$

$$\Omega_3 = 12\lambda i\gamma^2\varepsilon^2\delta_{-1,-1} r^3 \exp[i(\theta - \ell z)]$$

$$- 12\lambda i\gamma^2\varepsilon^2\delta_{1,-1} r^2 \exp[i(-\theta- \ell z)]$$

$$\Omega_4 = 48\lambda^2 i\gamma^2\varepsilon^2\delta_{1,-1} r^5 \exp[i(\theta - \ell z)]$$

For the singular $n = \pm 2$ solutions:

$$\underset{\sim}{Q}_0 = -2i\gamma\varepsilon(\ell + 2)\delta_{2,\ell}\, r \exp[i(2\theta - \ell z)](i\hat{r} - \hat{\theta})$$

$$\underset{\sim}{Q}_1 = 0 \tag{48}$$

$$\underset{\sim}{Q}_2 = 2i\gamma\varepsilon\delta_{2,\ell} \exp[i(2\theta - \ell z)](i\hat{r} - \hat{\theta})$$

$$+ 2i\gamma\varepsilon\delta_{2,\ell}\, r \exp[i(-2\theta - \ell z)](i\hat{r} + \hat{\theta}) \quad ,$$

whereas, Ω_0, Ω_1 are non-singular and

$$\Omega_2 = -i\gamma^2\varepsilon^2(\ell + 2)\delta_{2,\ell}\, r^2 \exp[i(2\theta - \ell z)]$$

$$- i\gamma^2\varepsilon^2(\ell - 2)\delta_{2,\ell}\, r^2 \exp[-i(2\theta + \ell z)] \tag{49}$$

$$\Omega_3 = -2i\gamma^2\varepsilon^2\delta_{2,\ell} r^2 \exp[-i\ell z]$$

In the last section we computed the equilibrium fields for both the plasma and vacuum regions. In this section we have used these results together with the Euler equations to evaluate ξ, $\underset{\sim}{Q}$, and Ω. We now use these results to compute δW^P, δW^Σ, and δW^V and finally determine δW and its unstable stationary states to leading order in c.

$\underline{\delta W^P}$:

We express δW^P as the sum of three terms

$$\delta W = \delta W_1^P + \delta W_2^P + \delta W_3^P$$

where each term is computed as follows:

In view of (23)', making use of (40), (42), (43), (45), (10') and (20'), we have for those coefficients, $\delta_{n,\ell}$, for which $n \neq \pm 1, \pm 2$,

$$\delta W_1^p = \int r(\bar{n}\cdot(\underset{\sim}{\xi}_o + \underset{\sim}{\xi}_1 + \underset{\sim}{\xi}_2))\ (\bar{\Omega}_o + \bar{\Omega}_1 + \bar{\Omega}_2)d\theta dz$$

$$= 2\pi\gamma^2\varepsilon^2 H \sum_{n>2} (n)\left((\ell + n) - \frac{nc^2}{2}\right)^2 \delta_{n,\ell}^2 \qquad (50)$$

while from (40), (42), and (49), for $\delta_{\pm 2,\ell}$

$$\delta W_2^p = 2\pi H\gamma^2\varepsilon^2 \left\{ 2\left((\ell+2)-c^2\right)^2\delta_{2,\ell} + 2\left((\ell-2)+c^2\right)^2\delta_{-2,\ell} + 2c^4\delta_{2,\ell}^2 + 2c^4\delta_{-2,\ell}^2 \right\} . \qquad (51)$$

Finally, from (40) and (46), taking into account leading order mixed terms containing the coefficients, $\delta_{\pm 1,\ell}$ $\delta_{\pm 3,\ell}$

$$\delta W_3^p = 2\pi H\gamma^2\varepsilon^2\left((\ell+1)\delta_{1,\ell}+c\delta_{-1,\ell}\right)^2 + \left((\ell-1)\delta_{-1,\ell}-c\delta_{1,\ell}\right)^2 + 24\left((\ell-3)\delta_{-3,\ell}+4\lambda c^2\delta_{1,\ell}\right)^2 + 24\left((\ell+3)\delta_{3,\ell}+4\lambda c^2\delta_{-1,\ell}\right)^2 \qquad (52)$$

$\underline{\delta W^\Sigma}$:

From (22) and $(\bar{n}\cdot\underset{\sim}{\xi})^2 = (\xi^r)^2 + n_\theta^2(\xi^\theta)^2 = (\xi_o^r+\xi_1^r)^2 + O(c^2)$ where $\bar{n}$ is given by (20'), as $\bar{n} = \hat{r} + n_\theta\hat{\theta}$ we find to leading order in c

$$\delta W^\Sigma = 2\pi H\gamma^2\varepsilon^2 \left\{ - \sum_{n\neq o} \frac{n^2c^2}{2}\beta\delta_{n,\ell}^2 - \sum_{n>1} 4n(n+2)c^2\beta\delta_{n,\ell}\delta_{n+2,\ell} - \sum_{n<-1} 4n(n-2)c^2\beta\delta_{n,\ell}\delta_{n-2,\ell} - \sum_{\substack{\ell+n\neq o \\ n>1}} \frac{n^3(n+2)c^2\beta\delta_{n,\ell}\delta_{n+2,\ell}}{\ell+n} - \sum_{\substack{\ell+n\neq o \\ n<-1}} \frac{n^3(n-2)c^2\beta\delta_{n,\ell}\delta_{n-2,\ell}}{\ell+n} + 2c\beta\delta_{-1,\ell}\delta_{1,\ell} \right\} \qquad (53)$$

$\underline{\delta W^V}$:

The magnetic potential, Φ, satisfies the Euler equation (28), and the boundary condition (26)

$$\underset{\sim}{\bar{n}} \cdot \nabla \underset{\sim}{\Phi} = \underset{\sim}{\bar{n}} \cdot \left(\nabla \times \left(\underset{\sim}{\xi} \times \underset{\sim}{B}^{out} \right) \right)$$

at the plasma vacuum interface and,

$$\frac{\partial \Phi}{\partial n} = 0$$

at the outer flux surface.

The outer boundary of the vacuum region, i.e., the conducting surface, should be taken to be a flux surface, determined by the condition $\psi(r,\theta)$ given by (8) is a constant. However, as suggested by Weitzner (Ref. 7), to simplify the computations, we approximate the outer conductor by the cylindrical surface $r = \rho$, and thus

$$\frac{\partial \Phi}{\partial r} (\rho) = 0 \quad . \tag{54}$$

Since the boundary condition on the conducting surface is the natural boundary condition for (28), we underestimate (overestimate) the stability of the system by taking ρ so that the true vacuum region lies outside (inside) the surface $r(\theta) = \rho$. In this manner we can readily verify the insensitivity of the stability predictions to the approximate outer conductor radius, ρ. Thus, if we show that the system is stable for ρ less than some critical value ρ_c, the actual value of ρ_c can be taken to be somewhat larger.

We proceed with the computation of δW^V. From (28) and (54),

$$\Phi = \sum_{n>o} \beta_n (r^{-n} + r^n \rho^{-2n}) \exp[i(n\theta - \ell z)]$$

$$+ \sum_{n<o} \beta_n (r^n + r^{-n}\rho^{2n}) \exp[i(n\theta - \ell z)] \qquad (55)$$

and,

$$\underset{\sim}{\hat{n}} \cdot \underset{\sim}{\nabla} \Phi = \sum_{n>o} n\beta_n (-1 + r^{2n}\rho^{-2n}) r^{-n-1} \exp[i(n\theta - \ell z)]$$

$$+ \sum_{n<o} n\beta_n (1 - r^{-2n}\rho^{2n}) r^{-n-1} \exp[i(n\theta - \ell z)] \qquad (56)$$

From (40), (42), (43), and (15), to leading order in c:

$$\underset{\sim}{\hat{n}} \cdot \left(\underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B}^{out}) \right) = \zeta\varepsilon \sum_{n>1} n\delta_{n,\ell} r^{n-1} \left((\ell+n) - \frac{nc^2}{2} \right) \exp[i(n\theta - \ell z)]$$

$$+ \left((\ell+1)\delta_{1,\ell} + c\delta_{-1,\ell} \right) \exp[i(\theta - \ell z)]$$

$$+ \left((\ell-1)\delta_{-1,\ell} - c\delta_{1,\ell} \right) \exp[i(-\theta - \ell z)]$$

From boundary condition (26), solving order by order in c:

$$\beta_n = \zeta\varepsilon k_n \left(\frac{nc^2}{2} - (\ell + n) \right) \delta_{n,\ell} \quad , \qquad n > 1$$

$$\beta_1 = -\zeta\varepsilon k_1 \left(c\delta_{-1,\ell} + (\ell + 1)\delta_{1,\ell} \right) \qquad (57)$$

$$\beta_{-1} = \zeta\varepsilon k_1 \left(c\delta_{1,\ell} - (\ell - 1)\delta_{-1,\ell} \right)$$

where,

$$k_n = \frac{\rho^{2n}}{\rho^{2n}_{-1}}$$

Thus, from (55), (56), and (25)', where $\hat{n} = (1 - \frac{c^2}{4})\hat{r}$ + angular dependence,

$$\delta W^V = 2\pi H\gamma^2 \varepsilon^2 \left\{ \sum_{n>1} n L_n \delta^2_{n,\ell} \left((\ell+n) - \frac{nc^2}{2} \right)^2 + L_1 \left((\ell+1)\delta_{1,\ell} + c\delta_{-1,\ell} \right)^2 \right.$$

$$\left. + L_1 \left(-c\delta_{1,\ell} + (\ell-1)\delta_{-1,\ell} \right)^2 + 2L_2 c^4 \delta^2_{2,\ell} + 2L_2 c^4 \delta^2_{-2,\ell} \right\} \qquad (58)$$

where $L_n = k_n(1 + \rho^{-2n})$.

$\underline{\delta W}$:

Combining (50), (51), (52), (53) and (58) assuming that β is of order c^2

$$\delta W = 2\pi H\gamma^2\varepsilon^2 \sum_{n\neq o,\pm 1,\pm 3} \Big\{ (1 + L_n)n\big((\ell + n) - \frac{nc^2}{2}\big)^2 \delta^2_{n,\ell}$$

$$+ 2(1 + L_2)c^4\delta^2_{2,\ell} + 2(1 + L_2)c^4\delta^2_{-2,\ell}$$

$$+ (1+L_1)\big\{(\ell+1)\delta_{1,\ell}+c\delta_{-1,\ell}\big\}^2 + (1+L_1)\big\{(\ell-1)\delta_{-1,\ell}-c\delta_{1,\ell}\big\}^2$$

$$+ 2c\beta\delta_{-1,\ell}\delta_{1,\ell} - \sum_{n\neq o,\pm 1} n^2\,\frac{c^2}{2}\,\beta\delta^2_{n,\ell}$$

$$+ 24(1 + L_3)\big\{(\ell - 3)\delta_{-3,\ell}+ 4\lambda c^2\delta_{1,\ell}\big\}^2$$

$$+ 24(1 + L_3)\big\{(\ell + 3)\delta_{3,\ell} + 4\lambda c^2\delta_{-1,\ell}\big\}^2 \Big\} \qquad (59)$$

Note that for $\ell + n = O(1)$, the non-singular diagonal terms (59) will dominate unless $\delta_{n,\ell} = O(c^2)$. Thus, since we are interested in determining the unstable states, we set $\delta_{n,\ell} = O(c^2)$ when $\ell + n = O(1)$, in which case the mixed terms in (53) are negligible and are not included in (59).

It should also be noted that the leading order components of the eigenstates of (59), corresponding to arbitrary ℓ and $n < 0$, can be identified with complex conjugate representations of the same state with $n > 0$. Thus, for $n \neq \pm 1, \pm 3$ we need only consider states with $n > 0$.

Finally, if the various components of the displacement, $\underset{\sim}{\xi}$, are rewritten in the physically meaningful, cylindrical coordinates, the actual wavenumbers ℓ', n', are given in

terms of ℓ and n as

$$\ell' = \frac{\varepsilon}{r_o}(\ell + n)$$

$$n' = n$$

If we relate ℓ' to the height of the plasma column, i.e.,

$$\ell' = \frac{2\pi}{H}$$

from (59) each value of n corresponds to a family of unstable eigenstates under the conditions,

$$\frac{2\pi r_o}{\varepsilon\left(\frac{n}{2}c^2+c\sqrt{\frac{n\beta}{2(1+L_n)}}\right)} < H < \frac{2\pi r_o}{\varepsilon\left(\frac{n}{2}c^2-c\sqrt{\frac{n\beta}{2(1+L_n)}}\right)}, \quad n > 2, \text{ for all } \beta$$

$$\frac{2\pi r_o}{\varepsilon\left(c^2+c\sqrt{\frac{\beta}{1+L_2}-c^2}\right)} < H < \frac{2\pi r_o}{\varepsilon\left(c^2-c\sqrt{\frac{\beta}{1+L_2}-c^2}\right)}, \quad n=2, \quad \beta \geq (1+L_2)c^2 \tag{60}$$

$$\frac{2\pi r_o}{\varepsilon\left(\frac{c^2}{2}+c\sqrt{\frac{\beta}{1+L_1}}\right)} < H < \frac{2\pi r_o}{\varepsilon\left(\frac{c^2}{2}-c\sqrt{\frac{\beta}{1+L_1}}\right)}, \quad n = 1, \text{ for all } \beta,$$

$$\text{and } \delta_{-1,\ell} = -\frac{c}{2}\,\delta_{1,\ell}$$

Minimizing δW with respect to c for each leading order n,ℓ' mode of the above families:

$$\delta W(n,\ell') = \begin{cases} \left(-\frac{n\beta^2}{4(1+L_n)} - \frac{r_o}{\epsilon} n\beta\ell'\right)\delta^2_{n,\ell'}, & n > 2, \text{ for all } \beta \\ \left[(1+L_2)\left(\frac{r_o\ell'}{\epsilon} - \frac{\beta}{2(1+L_2)}\right)^2 - \frac{5\beta^2}{8(1+L_2)}\right]\delta^2_{2,\ell} & \\ \qquad , \; n=2, \quad \beta \geq (1+L_2)c^2 & \\ \left(-\frac{\beta^2}{1+L_1} - \frac{2r_o}{\epsilon}\beta\ell'\right)\delta^2_{1,-1}, & n = 1, \text{ for all } \beta, \\ \qquad \delta_{-1,\ell} = -\frac{c}{2}\delta_{1,\ell} & \end{cases}$$

$$2\pi H\gamma^2\epsilon^2$$

These instabilities were observed by Freidberg (Ref. 4) in his study of helically symmetric systems at high betas. They are referred to as "kink instabilities."

If one chooses the column length to avoid kink instabilities for some range of n values for which the theory applies, we see from (60) that there exists a critical beta, $\beta_c = \frac{(L_1+1)}{4} c^2$, and corresponding to β_c, there exists a marginally stable state, $\bar{\xi}^{(0)}$. The coefficients of the leading order terms of $\bar{\xi}^{(0)}$ are

$$\delta_{-1,-1} = -\, c/2\ \delta_{1,-1}$$

$$\delta_{-3,-1} = \lambda c^2\ \delta_{1,-1}$$

$$\delta_{3,-1} = -\lambda c^3\ \delta_{1,-1}$$

Since δW is a homogeneous quadratic form, we can set $\delta_{1,-1} = 1$ in which case

$$\underset{\sim}{\xi}^{(0)} = \exp[i(\theta + z)](i\hat{r} - \hat{\theta})\,\frac{c}{2}\exp[i(-\theta + z)](i\hat{r} + \hat{\theta})$$
$$- 3\lambda c^2 r^2 \exp[i(-3\theta + z)](i\hat{r} + \hat{\theta})$$
$$- 3\lambda c^3 r^3 \exp[i(3\theta + z)](i\hat{r} - \hat{\theta}) \qquad (61)$$

Corresponding to $\underset{\sim}{\xi}^{(0)}$, we state for future reference, to leading order in c,

$$\underset{\sim}{\Omega}^{(0)} = i\gamma\varepsilon\,\frac{c^2}{2}\exp[i(\theta + z)](i\hat{r} - \hat{\theta}) \qquad (62)$$

and

$$\Omega^{(0)} = i\gamma\varepsilon\,\frac{c^3}{2}\,r\exp[i(-\theta + z)] \qquad (63)$$

$$\phi^{(0)} = \gamma\varepsilon k_1\,\frac{c^2}{2}(r^{-1} + r\rho^{-2})\exp[i(\theta + z)] \qquad (64)$$

Bifurcated Equilibrium

In the above stability analysis, we found a family of marginally stable states, $\underset{\sim}{\xi}^{(o)}$, given by (61) corresponding to a critical beta, $\beta_c = (L_1+1)/4\ c^2$. In this section, we will determine bifurcated equilibrium states in a neighborhood of the original equilibria.

In an exact bifurcation analysis, the bifurcated equilibria must satisfy the same flux constraints as the original equilibria. Although we will not maintain these constraints exactly, they will hold to leading orders in the expansion parameters.

To determine the bifurcated equilibria, we choose one of the marginally stable states, $\underset{\sim}{\xi}^{(o)}$, by arbitrarily fixing the value of c, and introducing another small parameter, d. We then expand the fields, $\underset{\sim}{B}$, plasma surface radius, r, and plasma surface normal, $\hat{n}$, with respect to d. Specifically we set

$$\begin{aligned} \underset{\sim}{B} &= \underset{\sim}{B}^{(o)} + d\underset{\sim}{B}^{(1)} + d^2\underset{\sim}{B}^{(2)} + \ldots \\ r &= r^{(o)} + dr^{(1)} + d^2r^{(2)} + \ldots \\ \hat{n} &= \hat{n}^{(o)} + d\hat{n}^{(1)} + d^2\hat{n}^{(2)} + \ldots \end{aligned} \tag{65}$$

where $\underset{\sim}{B}^{(o)}$, $r^{(o)}$, and $\hat{n}^{(o)}$ correspond to the bifurcating state. The first order term $r^{(1)}$ is the projection of the displacement, $\underset{\sim}{\xi}^{(o)}$, onto the surface normal, $\hat{n}^{(o)}$, i.e., $r^{(1)} = \hat{n}^{(o)}\cdot\underset{\sim}{\xi}^{(o)}$. But $\hat{n}^{(o)}\cdot\underset{\sim}{\xi}^{(o)}$ is a complex vector, since $\underset{\sim}{\xi}$ has been expressed, for computational convenience, in

complex form. Thus, we may, with all generality, set

$$r^{(1)} = \mathrm{Im}\ (\hat{n}^{(o)} \cdot \underset{\sim}{\overline{\xi}}^{(o)})$$

Thus, from (20') and (61)

$$r^{(1)} = \cos(\theta + z) + \frac{c}{2}\cos(3\theta + z) + \left(\frac{3}{8} - 2\lambda\right)c^2 \cos(5\theta+z) + \left(\frac{1}{8} - \lambda\right)c^2 \cos(3\theta - z) \tag{66}$$

Once $r^{(1)}$ is determined, $\hat{n}^{(1)}$ can be computed from (20)

$$\hat{n}^{(1)} = \sin(\theta + z)\hat{\theta} - \varepsilon c \sin(3\theta + z)\hat{z} \tag{67}$$

Finally, the first order perturbation of $\underset{\sim}{B}$, or first variation of $\underset{\sim}{B}^{(o)}$, can be shown to be $\underset{\sim}{Q}^{(o)}$. Thus, from (62),

$$\underset{\sim}{B}^{(1)}_{in} = \mathrm{Im}\ \underset{\sim}{\overline{Q}}^{(o)in} = -\gamma\varepsilon\frac{c^2}{2}\left(\sin(\theta + z)\hat{r} + \cos(\theta + z)\hat{\theta}\right) \tag{68}$$

whereas, from (64)

$$\underset{\sim}{B}^{(1)out} = \mathrm{Im}\ \underset{\sim}{\nabla\overline{\Phi}} = \zeta\varepsilon\frac{c^2}{2} r^{-2}\left(-\sin(\theta+z)\hat{r}+L_1 \cos(\theta+z)\hat{\theta}\right) \tag{69}$$

We note that to order d,

$$[\underset{\sim}{B}^2] = [\underset{\sim}{B}^{(o)2}] + 2d[\underset{\sim}{B}^{(o)} \cdot \underset{\sim}{B}^{(1)}] \quad .$$

Since $\beta_c = (L_1+1)/4\ c^2$ from (22'),

$$[\underset{\sim}{B}^{(o)2}] = \zeta^2\varepsilon^2c^3\left(\frac{L_1+1}{4}\right)(-r^2 + r^{-2})\cos 2\theta \tag{70}$$

while from (9'), (15), (68), and (69):

$$2d[\underset{\sim}{B}^{(o)} \cdot \underset{\sim}{B}^{(1)}] = \gamma^2\varepsilon^2c^3d\left\{ r^{-1}\left(\frac{L_1+1}{2}\right)\cos(3\theta + z) + r^{-1}\left(\frac{L_1-1}{2}\right)\cos(\theta - z) + r\cos(\theta - z)\right\} . \tag{71}$$

From (70) and (71), we verify

$$[\underset{\sim}{B}^2] = \text{constant}$$

$$\hat{n}\cdot\underset{\sim}{B} = 0$$

to order d and leading order in c, which is a necessary condition for the bifurcated fields to constitute a family of equilibria.

We complete the equilibrium analysis by determining the d^2 terms, $\underset{\sim}{B}^{(2)}$, $r^{(2)}$, and $\hat{n}^{(2)}$ of the bifurcated expansion, (65). It is at this point that we may deviate from the exact flux constraints, since no constraints are imposed.

From (20), and in terms of the unknown $r^{(2)}$,

$$\hat{n}^{(2)} = \left(-\frac{1}{4} + \frac{1}{4}\cos 2(\theta+z)\right)\hat{r} + \left(-\frac{1}{2}\sin 2(\theta+z) - \frac{\partial r^{(2)}}{\partial\theta}\right)\hat{\theta}$$

$$+ \varepsilon\left(\frac{\partial}{\partial\theta} - \frac{\partial}{\partial z}\right)r^{(2)}\hat{z} \quad . \tag{72}$$

In view of the above equations for r, $\hat{n}$, and $\underset{\sim}{B}$, we can, with all generality, set

$$\underset{\sim}{B}^{(2)r}_{in} = \gamma\varepsilon cr^3 c_1 \sin(4\theta + 2z)$$

$$+ \gamma\varepsilon crc_2 \sin 2z + \gamma\varepsilon crc_3 \sin 2\theta \tag{73}$$

while from (19),

$$\underset{\sim}{B}^{(2)z}_{in} = -\frac{\gamma\varepsilon^2 cr^3 c_1}{2}\cos(4\theta + 2z) \quad . \tag{73'}$$

Also, we can set,

$$r^{(2)} = \alpha_2 + \alpha_1 \cos 2(\theta+z) + \alpha_3 c \sin 2\theta \sin 2(\theta+z)$$

$$+ \alpha_4 c \cos 2\theta \cos 2(\theta + z) \quad ,$$

where α_1, α_2, α_3, α_4, c_1 are determined from the equilibrium conditions, $[\underset{\sim}{B}^2]$ = constant and $\hat{n} \cdot \underset{\sim}{B} = 0$.

In view of the above equations, the d^2 terms of $\hat{n} \cdot \underset{\sim}{B}$ result in,

$$\hat{n}\cdot\underset{\sim}{B} = 0 = \gamma\epsilon c d^2[(-\tfrac{1}{4} + \alpha_2) \sin 2\theta + (\tfrac{1}{4} + \alpha_1 - 2\alpha_4 + c_1) \sin 2\theta \cos 2(\theta + z) + (2\alpha_1 + 2\alpha_3 + c_1) \cos 2\theta \sin 2(\theta + z)] \quad (74)$$

Next, since to leading order in c,

$$[\underset{\sim}{B}^2] = [\underset{\sim}{B}^{(o)2}] + 2d[\underset{\sim}{B}^{(o)}\cdot\underset{\sim}{B}^{(1)}] + 2d^2[B^{(o)}z_B{}^{(2)}z] + d^2[\underset{\sim}{B}^{(1)2}]$$

From (70), (71), and (73), to leading order in c,

$$[\underset{\sim}{B}^2]=\text{constant}=\zeta^2\varepsilon^2c^3d^2\Big\{(\tfrac{L_1+1}{2})(\tfrac{1}{2} -2\alpha_1- \tfrac{c_1}{2}) \cos 2\theta \cos 2(\theta+z) + (\tfrac{L_1+1}{2})(1+ \tfrac{c_1}{2}) \sin 2\theta \sin 2(\theta+z) \Big\} \quad (75)$$
$$+ (\tfrac{L_1+1}{2})(c_2+c_3) \sin (\theta+z) \sin (\theta-z)$$
$$+ \{(\tfrac{L_1+1}{2})(c_2-c_3) - (\tfrac{L_1-1}{2})\} \cos(\theta+z)\cos(\theta-z)$$

From (68) and (69), $[B^{(1)2}]$ is found to be of smaller order in c and, therefore, does not enter the present computation, although it is stated for future reference

$$[\underset{\sim}{B}^{(1)2}] = \zeta^2\varepsilon^2 \tfrac{c^4}{4} [r^{-4}(1 + L_1^2) - 2] \quad (76)$$

Finally, from (74) and (75),

$$c_1 = -2$$

$$\alpha_1 = 3/4$$

$$\alpha_2 = \alpha_3 = 1/4$$

$$\alpha_4 = -1/2$$

$$c_2 = -c_3 = \frac{1}{2}\left(\frac{L_1-1}{L_1+1}\right)$$

Thus, we have succeeded in determining the d^2 component of the bifurcated equilibria which we summarize as

$$r^{(2)} = \frac{1}{4} + \frac{3}{4}\cos 2(\theta + z)$$

$$+ \frac{c}{4}\sin 2\theta \sin 2(\theta + z) - \frac{c}{2}\cos 2\theta \cos 2(\theta + z)$$

$$\underset{\sim}{B}^{(2)} = -2\gamma\varepsilon cr^3\left[\sin(4\theta + 2z)\hat{r} + \cos(4\theta + 2z)\hat{\theta}\right]$$

$$+ \frac{1}{2}\gamma\varepsilon cr\left(\frac{L_1-1}{L_1+1}\right)(\sin 2z\hat{r} + \cos 2z\hat{\theta}) \quad . \tag{77}$$

It should be noted in (78), the $(\sin 2\theta\hat{r} + \cos 2\theta\hat{\theta})$ term has been absorbed into $\underset{\sim}{B}^{(o)}$ by redefining the arbitrary constant, c, as $c + cd^2 \equiv c'$.

Finally, we summarize for future reference,

$$r = 1 + \frac{d^2}{4} + \frac{c}{2}\cos 2\theta + \left(\frac{3}{16} - \lambda\right)c^2\cos 4\theta + d\cos(\theta + z)$$

$$+ \frac{cd}{2}\cos(3\theta + z) + \left(\frac{3}{8} - 2\lambda\right)c^2 d\cos(5\theta + z) \tag{78}$$

$$+ \left(\frac{1}{8} - \lambda\right)c^2 d\cos(3\theta - z)$$

$$+ (\text{terms with axial dependence } 2z) \quad .$$

From (78) it follows, making use of (20), that

$$\hat{n} = \left(1 - \frac{d^2}{4} + \frac{cd}{2}\cos(3\theta+z) - \frac{cd}{2}\cos(\theta-z) - \frac{cd^2}{2}\cos 2\theta\right)\hat{r}$$
$$+ \left((c - \frac{3}{4}cd^2)\sin 2\theta + d\sin(\theta + z)\right.$$
$$\left. + \frac{3}{4}cd\sin(3\theta + z) - \frac{cd}{4}\sin(\theta - z)\right)\hat{\theta} \qquad (79)$$
$$+ (\text{terms with axial dependence } 2z)$$

Finally, making use of (70), (71), (76), (77) and (79), we state for future reference

$$\hat{n}\cdot\nabla\frac{[B^2]}{2} = \zeta^2\varepsilon^2c^3\left\{\left[(\frac{L_1+1}{8})c + (\frac{11}{32}(L_1+1) - (\frac{L_1^2+1}{2}))cd^2\right]\right.$$
$$+ \left[-(\frac{L_1+3}{4})d^2 - (\frac{L_1+1}{2})\right]\cos 2\theta$$
$$- (\frac{3L_1+5}{8})cd\cos(\theta + z) + d\cos(\theta - z) \qquad (80)$$
$$\left. + (\text{terms having axial dependence } 2z)\right\}$$

Bifurcated Stability

In order to study the stability of the bifurcated equilibria, we must again determine the sign of the second variation of δW for these new equilibria.

In the prebifurcated stability analysis, the leading order instability was due to the singular eigenstates, $n = \pm 1$, $\ell = -1$ to leading order in c. In this section, we will show that when the conducting wall is sufficiently far from the plasma vacuum interface, once again, the leading order instabilities will be due to the singular eigenstates $n = \pm 1$, $\ell = -1$; $n = 2$, $\ell = -2$; and $\ell+n = \pm 4$, $\ell = 0,2$ to leading order in c. All $\underset{\sim}{\xi}$ that are non-singular to order c^2 have been found to be stable.

In this section we will also show that these instabilities can be stabilized by placing the conducting wall within $1.15\ r_o$ of the plasma vacuum surface.

The analysis proceeds as before. Stationary values of δW are determined for admissible disturbances, $\underset{\sim}{\xi}$, satisfying the Euler-Lagrange equations. However, here we expand $\underset{\sim}{\xi}$, Ω, and $\underset{\sim}{Q}$ with respect to the parameter, d, in addition to c. We set

$$\underset{\sim}{\xi} = \underset{\sim}{\xi}^{(o)} + d\underset{\sim}{\xi}^{(1)} + d^2\underset{\sim}{\xi}^{(2)} + \ldots$$

$$\Omega = \Omega^{(o)} + d\Omega^{(1)} + d^2\Omega^{(2)} + \ldots$$

$$\underset{\sim}{Q} = \underset{\sim}{Q}^{(o)} + d\underset{\sim}{Q}^{(1)} + d^2\underset{\sim}{Q}^{(2)} + \ldots$$

where,

$$\underset{\sim}{\xi}^{(k)} = \underset{\sim}{\xi}_o^{(k)} + c\underset{\sim}{\xi}_1^{(k)} + c^2\underset{\sim}{\xi}_2^{(k)} + \ldots$$

$$\Omega^{(k)} = \Omega_o^{(k)} + c\Omega_1^{(k)} + c^2\Omega_2^{(k)} + \ldots$$

$$\underset{\sim}{Q}^{(k)} = \underset{\sim}{Q}_o^{(k)} + c\underset{\sim}{Q}_1^{(k)} + c^2\underset{\sim}{Q}_2^{(k)} + \ldots$$

for k = 0,1,2... and $\underset{\sim}{\xi}^{(o)}$, $\underset{\sim}{Q}^{(o)}$, and $\Omega^{(o)}$ correspond to the bifurcating state given by (61), (62), and (63), and as before denoted by $\overline{\underset{\sim}{\xi}}^{(o)}$, $\overline{\underset{\sim}{Q}}^{(o)}$, and $\overline{\Omega}^{(o)}$ to distinguish them from the general prebifurcated values, without superscripts given by (40), (42), (43), (44), (45), etc.

As before, $\underset{\sim}{Q}_j^{(k)}$ is expressed by (37), in terms of the unknown disturbances, $\underset{\sim}{\xi}_j^{(k)}$, and the bifurcated fields given by (9'), (68), and (77).

If we set

$$\underset{\sim}{Q}_{ij}^{(\ell k)} = (\underset{\sim}{B}_i^{(\ell)}\cdot\underset{\sim}{\nabla})\underset{\sim}{\xi}_{j-i}^{(k-\ell)} - (\underset{\sim}{\xi}_{j-i}^{(k-\ell)}\cdot\underset{\sim}{\nabla})\underset{\sim}{B}_i^{(\ell)}$$

then $\underset{\sim}{Q}_i^{(k)}$ is expressed as

$$\underset{\sim}{Q}_i^{(k)} = \sum_{\ell=o}^{k}\ \sum_{i=o}^{j} \underset{\sim}{Q}_{ij}^{(\ell k)}$$

where,

$$\underset{\sim}{B}_i^{(\ell)} = 0 \qquad \text{for } i,\ell > 2 \quad .$$

We now solve the Euler equation (38), for the components of $\underset{\sim}{\xi}$ arising from the bifurcated fields $\underset{\sim}{B}^{(1)}$ and $\underset{\sim}{B}^{(2)}$. These components arise in two ways: from the interaction of the bifurcated fields first with the bifurcating state $\overline{\underset{\sim}{\xi}}^{(o)}$, and then with the, as yet, undetermined order d components of the unstable eigenstates denoted by $\overline{\underset{\sim}{\xi}}^{(1)}$. We proceed with this analysis.

Due to the bifurcating state, $\overline{\underset{\sim}{\xi}}^{(o)}$,

$$\underset{\sim}{Q}_{jk}^{(ii)} = (\underset{\sim}{B}^{(i)}\cdot\underset{\sim}{\nabla})\overline{\underset{\sim}{\xi}}^{(o)} - (\overline{\underset{\sim}{\xi}}^{(o)}\cdot\underset{\sim}{\nabla})\underset{\sim}{B}^{(i)} \quad , \qquad i = 1,2$$

In view of (38) and making use of (9'), (61), (62), (68) and (77), we find that due to the interaction of the bifurcated fields with $\underset{\sim}{\bar{\xi}}^{(o)}$,

$$\nabla x \left\{ \underset{\sim}{B}^{(i)} \cdot \nabla \underset{\sim}{\bar{Q}}^{(o)} + \underset{\sim}{\bar{Q}}^{(o)} \cdot \nabla \underset{\sim}{B}^{(i)} + \underset{\sim}{B}^{(o)} \cdot \nabla \underset{\sim}{Q}_{jk}^{(ii)} + \underset{\sim}{Q}_{jk}^{(ii)} \cdot \nabla \underset{\sim}{B}^{(o)} \right\} = 0 \qquad i = 1,2$$

to order c^4 from which we conclude that $\underset{\sim}{\xi}^{(1)}$ and $\underset{\sim}{\xi}^{(o)}$ satisfy the same equations and thus have the same form up to order c^4. However, before we can conclude that $\underset{\sim}{\xi}^{(2)}$ and $\underset{\sim}{\xi}^{(o)}$ also have the same form, we must determine $\underset{\sim}{\bar{\xi}}^{(1)}$ and its interaction with $\underset{\sim}{B}^{(1)}$. We will carry this out in the Appendix and show that $\underset{\sim}{\xi}^{(2)} = \underset{\sim}{\xi}^{(o)}$.

We determine $\underset{\sim}{\bar{\xi}}^{(1)}$ to leading order in c by determining the unstable stationary states of $\delta W(\underset{\sim}{\xi})$ with respect to $\underset{\sim}{\xi} = \underset{\sim}{\bar{\xi}}^{(o)} + d\underset{\sim}{\xi}^{(1)}$. This is equivalent to diagonalizing the leading order terms of the quadratic form $\delta W(\underset{\sim}{\xi})$.

We find it useful to decompose $\delta W(\underset{\sim}{\xi})$ into the sum of four quadratic terms

$$\delta W(\underset{\sim}{\xi}) = \delta W(\underset{\sim}{\bar{\xi}}^{(o)}, \underset{\sim}{\bar{\xi}}^{(o)}) + \delta W(\underset{\sim}{\bar{\xi}}^{(o)}, \underset{\sim}{\bar{\xi}}^{(1)}) + \delta W(\underset{\sim}{\xi}^{(1)}, \underset{\sim}{\bar{\xi}}^{(o)}) + \delta W(\underset{\sim}{\xi}^{(1)}, \underset{\sim}{\xi}^{(1)})$$

These quadratics are further subdivided into two terms:

$$\delta W^{(o)} = \delta W_{oo}$$

and

$$\delta W^{(1)} = \delta W_{01} + \delta W_{10} + \delta W_{11}$$

where we have used the notation

$$\delta W_{ij} = \delta W(\xi^{(i)}, \xi^{(j)}) \qquad i,j = 0,1 \qquad (81)$$

First we compute $\delta W^{(o)}$ to leading order in c and d.

$\underline{\delta W^{(o)\Sigma}}$:

Since

$$\delta W^{(o)\Sigma} = \int \frac{r}{|\bar{n}|} (\underset{\sim}{\bar{n}}\cdot\underset{\sim}{\bar{\xi}}^{(o)})^2 \hat{n}\cdot\nabla \frac{[B^2]}{2} \, d\theta dz \quad ,$$

making use of (80), we find after a long computation,

$$\delta W^{(o)\Sigma} = 2\pi H\zeta^2\varepsilon^2[-(\frac{1+L_1}{4})c^4 - \frac{c^4d^2}{2}(1+L_1^2)] \tag{82}$$

to leading order in c and d.

$\underline{\delta W^{(o)p}}$:

Noting that

$$\delta W^{(o)p} = \int r(\underset{\sim}{\bar{n}}\cdot\underset{\sim}{\bar{\xi}}_o)\bar{\Omega}^{(o)} d\theta dz \quad .$$

from (61) and (63)

$$\delta W^{(o)p} = 2\pi H\zeta^2\varepsilon^2(\frac{c^4}{4} + \frac{3}{8}c^4d^2) \quad . \tag{83}$$

$\underline{\delta W^{(o)V}}$:

Again we consider the conducting wall to be a cylinder of radius ρr_o in which case the vacuum magnetic potential satisfies boundary conditions (26) and (54) and is given to leading order by

$$\begin{aligned}\Phi^{(o)} &= \beta_1(r^{-1} + r\rho^{-2}) \exp[i(\theta + z)] \\ &\quad + \beta_2(r^{-2} + r^2\rho^{-4}) \exp[iz(\theta + z)]\end{aligned} \tag{84}$$

where

$$\underset{\sim}{\bar{n}}\cdot\underset{\sim}{Q} = \underset{\sim}{\bar{n}}\cdot\nabla\Phi \tag{26'}$$

and,

$$\bar{\underset{\sim}{n}}\cdot\nabla\underset{\sim}{\Phi}^{(o)} = \sum_{n>o} n\beta_n \Big[(-1 + r^{2n}\rho^{-2n}) + id\sin(\theta+z)(1 + r^{2n}\rho^{-2n}) \Big] r^{-n-1}\exp[i(n\theta - \ell z)]. \tag{85}$$

Letting

$$\underset{\sim}{Q} = \bar{\underset{\sim}{Q}}^{(o)} + \underset{\sim}{Q}^{(1)}$$

from (62)

$$\bar{\underset{\sim}{Q}}^{(o)} = i\gamma\varepsilon\frac{c^2}{2}\exp[i(\theta + z)](i\hat{r} - \hat{\theta}),$$

while using (61) and (69),

$$\underset{\sim}{Q}^{(1)} = \nabla\times(\bar{\underset{\sim}{\xi}}^{(o)} \times \underset{\sim}{B}^{(1)})$$

$$= i\gamma\varepsilon c^2 r^{-3}\exp[i2(\theta + z)](i\hat{r} + \hat{\theta})\left(\frac{L_1+1}{2}\right)$$

$$+ i\gamma\varepsilon\frac{c^2}{2} r^{-3}(L_1-1)\hat{\theta} \quad .$$

Thus,

$$\bar{\underset{\sim}{n}}\cdot\nabla\Phi^{(o)} = \bar{\underset{\sim}{n}}\cdot\underset{\sim}{Q} = \zeta\varepsilon c^2\left\{ \left(-\frac{1}{2} + \left(\frac{3L_1-1}{4}\right)d^2\right)\exp[i(\theta + z)] + \frac{d}{4} + \frac{d}{4}(1 - 2L_1)\exp[2i(\theta + z)] \right\} \tag{86}$$

From (84), (85), solving (26') order by order in d,

$$\beta_1 = \zeta\varepsilon\left\{ \frac{k_1c^2}{2} + \frac{k_1c^2d^2}{2}(L_2 - 1)(2k_1 - 1) \right\}$$

$$\beta_2 = \zeta\varepsilon c^2 d\,\frac{k_2}{2}(2k_1 - 1)$$

where k_n and L_n were defined as in (57) and (58). Since,

$$\delta W^{(o)}V = \int r|\bar{\underset{\sim}{n}}|\Phi^{(o)}(\hat{n}\cdot\nabla\underset{\sim}{\Phi}^{(o)})d\theta dz \quad ,$$

making use of (84), (85), and (79), we find after a long computation,

$$\delta W^{(o)V} = 2\pi H\zeta^2\varepsilon^2 \left\{ L_1 \frac{c^4}{4} + \frac{d^2c^4}{4} [2L_1^2(L_2 - 1) - L_1] \right\} \quad (87)$$

$\underline{\delta W^{(o)}}$:

From (82), (83), and (86)

$$\delta W^{(o)} = 2\pi H\zeta^2\varepsilon^2 \left\{ d^2c^4 \left(-\frac{1}{8} + \frac{1}{4} [2L_1^2(L_2 - 2) - L_1]\right) \right\} \quad (88)$$

The effect of the wall on $\delta W^{(o)}$ can be determined from (88) at once for the two limiting cases, as the wall is placed arbitrarily far from the plasma, and as the wall is placed arbitrarily close to the plasma surface, i.e. as $\rho \to \infty$ and $\rho \to 1$.

From their definitions, $\lim_{\rho\to\infty} k_1$, $\lim_{\rho\to\infty} k_2$, $\lim_{\rho\to\infty} L_1$, $\lim_{\rho\to\infty} L_2 = 1$. (89)

Thus from (88),

$$\lim_{\rho\to\infty} \delta W^{(o)} = 2\pi H\zeta^2\varepsilon^2 \left(-\frac{7}{8} c^4d^2\right) \quad . \quad (90)$$

On the other hand, since

$$\lim_{\rho\to 1} \frac{k_1}{k_2} = 2 \ , \qquad \lim_{\rho\to 1} k_1 = \infty$$

and,

$$L_1 = 2k_1 - 1 \quad , \qquad L_2 = 2k_2 - 1 \quad , \quad (91)$$

from (88), for $\rho \simeq 1$,

$$\delta W^{(o)} = 2\pi H\zeta^2\varepsilon^2 \left\{ \left(2k_1^3 + O(k_1^2)\right) c^4d^2 \right\} \quad . \quad (92)$$

$\underline{\delta W^{(1)}}$:

We have found that only components of $\underset{\sim}{\xi}^{(1)}$ with coefficients $\delta_{n,\ell}$ where $\ell + n = \pm 2, \pm 4, 0$ and $\ell = 0, -2$ result in unstable stationary states. The only $\delta_{n,\ell}$ with the above properties are $\delta_{4,-2}$, $\delta_{2,0}$, $\delta_{-2,0}$, $\delta_{6,-2}$, $\delta_{4,0}$, $\delta_{-2,-2}$, $\delta_{-4,0}$, and $\delta_{2,-2}$. We will show in Appendix A that compared with the magnitude of the instability associated with $\delta_{2,-2}$, the other instabilities are negligible. We will also show that although the presence of a conducting wall may lessen these instabilities, they cannot be removed. Moreover, we will show that the instability due to the leading order mode associated with $\delta_{2,-2}$ worsens as the conducting wall is brought closer to the plasma surface. In fact, the $\delta_{2,-2}$ interacts with $\overline{\underset{\sim}{\xi}}^{(o)}$ in such a way as to counteract most of the stabilizing effect of $\delta W^{(o)}$ in the presence of a conducting wall.

We proceed to determine $\delta W^{(1)}$ due to the leading order eigenmode associated with $\delta_{2,-2}$.

$\underline{\delta W^{(1)p}}$:

Using the notation (81),

$$\delta W^{(1)p} = \underset{01}{\delta W^{(1)p}} + \underset{10}{\delta W^{(1)p}} + \underset{11}{\delta W^{(1)p}}$$

where,

$$\delta W^{(1)p}_{01} = \int r(\underset{\sim}{\bar{n}}\cdot\underset{\sim}{\xi}^{(o)})\Omega^{(1)}d\theta dz$$

$$\delta W^{(1)p}_{10} = \int r(\underset{\sim}{\bar{n}}\cdot\underset{\sim}{\xi}^{(1)})\Omega^{(o)}d\theta dz$$

$$\delta W^{(1)p}_{11} = \int r(\underset{\sim}{\bar{n}}\cdot\underset{\sim}{\xi}^{(1)})\Omega^{(1)}d\theta dz$$

Since, as mentioned above, $\underset{\sim}{\xi}^{(1)}$, $\underset{\sim}{\xi}^{(o)}$ and $\Omega^{(1)}$, $\Omega^{(o)}$ have the same form, $\delta W^{(1)p}_{11}$ is identical with the prebifurcated value of δW^{p}. Thus according to (51),

$$\delta W^{(1)p}_{11} = 2\pi H\zeta^2\varepsilon^2(4c^4\delta^2_{2,-2}) \quad .$$

From (63) we find that

$$\delta W^{(1)p}_{10} = 2\pi H\zeta^2\varepsilon^2(\frac{c^4}{2}\, d\delta_{2,-2})$$

while from (61) and (49),

$$\delta W^{(1)p}_{01} = 2\pi H\zeta^2\varepsilon^2(\frac{17}{8}\, c^2 d\delta_{2,-2}) \quad .$$

Thus,

$$\delta W^{(1)p} = 2\pi H\zeta^2\varepsilon^2(4c^4\delta^2_{2,-2} + \frac{21}{8}\, c^2 d\delta_{2,-2}) \quad . \qquad (93)$$

$\underline{\delta W^{(1)\Sigma}}$:

From (21) and (24)

$$\delta W^{(1)\Sigma} = \int \frac{r}{|\underset{\sim}{\bar{n}}|}\,(\underset{\sim}{\bar{n}}\cdot\underset{\sim}{\xi})^2\,\hat{n}\cdot\nabla\,\frac{[B^2]}{2}\,d\theta dz$$

where,

$$(\underset{\sim}{\bar{n}}\cdot\underset{\sim}{\xi})^2 = (\underset{\sim}{\bar{n}}\cdot(\underset{\sim}{\xi}^{(o)} + \underset{\sim}{\xi}^{(1)}))^2 \quad . \qquad (94)$$

We again find it useful to decompose $\delta W^{(1)\Sigma}$ into

$$\delta W^{(1)\Sigma} = \delta W^{(1)\Sigma}_{01} + \delta W^{(1)\Sigma}_{10} + \delta W^{(1)\Sigma}_{11} \quad .$$

Once again $\delta W_{11}^{(1)\Sigma}$ is identical with the prebifurcated results, and referring to (53),

$$\delta W_{11}^{(1)\Sigma} = 2\pi H\zeta^2\varepsilon^2\left(-\frac{(L_1+1)}{2}c^4\delta_{2,-2}^2\right) \quad . \qquad (95)$$

From (80) and (94), we find after a long computation,

$$\delta W_{01}^{(1)\Sigma} + \delta W_{10}^{(1)\Sigma} = 2\pi H\zeta^2\varepsilon^2\left(-\frac{(7L_1+5)}{4}c^4 d\delta_{2,-2}\right) . \qquad (96)$$

$\underline{\delta W^{(1)V}}$:

We decompose $\delta W^{(1)V}$ into three terms:

$$\delta W^{(1)V} = \delta W_{01}^{(1)V} + \delta W_{10}^{(1)V} + \delta W_{11}^{(1)V}$$

where,

$$\delta W_{01}^{(1)V} = \int r|\underset{\sim}{\bar{n}}|\Phi^{(o)}\hat{n}\cdot\nabla\underset{\sim}{\Phi}^{(1)}d\theta dz$$

$$\delta W_{10}^{(1)V} = \int r|\underset{\sim}{\bar{n}}|\Phi^{(1)}\hat{n}\cdot\nabla\underset{\sim}{\Phi}^{(o)}d\theta dz$$

$$\delta W_{11}^{(1)V} = \int r|\underset{\sim}{\bar{n}}|\Phi^{(1)}\hat{n}\cdot\nabla\underset{\sim}{\Phi}^{(1)}d\theta dz$$

where $\Phi^{(o)}$ is given by (84), while $\Phi^{(1)}$ is the scalar potential that results from $\underset{\sim}{Q}^{(1)}$ and the boundary condition (26).

From (48),

$$\underset{\sim}{Q}^{(1)} = 2i\zeta\varepsilon c^2\delta_{2,-2}r\exp[2i(\theta+z)](i\hat{r}-\hat{\theta})$$
$$+ 2i\gamma\varepsilon c^2\delta_{2,-2}r\exp[2i(\theta+z)](i\hat{r}+\hat{\theta}) \quad .$$

Thus,

$$\bar{\underset{\sim}{n}}\cdot\nabla_{\underset{\sim}{Q}}\Phi^{(1)} = \bar{\underset{\sim}{n}}\cdot\underset{\sim}{Q}^{(1)} = 2\zeta\varepsilon c^2\delta_{2,-2}\left\{\exp\left[2i(\theta+z)\right] + \exp\left[2i(-\theta+z)\right]\right\} \quad . \tag{97}$$

From (85) and (97), solving (26') order by order,

$$\Phi^{(1)} = \zeta\varepsilon c^2\delta_{2,-2}\left\{\beta_3(r^{-2}+r^2\rho^{-4})\exp\left[2i(\theta+z)\right] + \beta_4(r^{-2}+r^2\rho^{-4})\exp\left[2i(-\theta+z)\right] + \beta_5(r^{-1}+r\rho^{-2})\exp\left[i(\theta+z)\right]\right\} \tag{98}$$

where,

$$\beta_3 = \beta_4 = \zeta\varepsilon k_2 c^2\delta_{2,-2}$$

$$\beta_5 = 2\zeta\varepsilon c^2 d\delta_{2,-2}k_1k_2$$

Thus, from (84), (86), (97), and (98),

$$\delta W^{(1)V}_{11} + \delta W^{(1)V}_{01} + \delta W^{(1)V}_{10} = \int r|\bar{n}|\left(\hat{n}\cdot\nabla_{\underset{\sim}{Q}}(\Phi^{(o)}+\Phi^{(1)})\right)(\Phi^{(o)}+\Phi^{(1)})\,d\theta dz \tag{99}$$

$$= 2\pi H\zeta^2\varepsilon^2\left(4L_2\delta^2_{2,-2}c^4 + L_1(2L_2+1)c^4 d\delta_{2,-2}\right)$$

$\underline{\delta W^{(1)}}$:

Adding (93), (95), (96) and (99) and diagonalizing the result,

$$\delta W^{(1)} = 2\pi H\zeta^2\varepsilon^2\left\{(8k_2-k_1)c^4\left[\delta_{2,-2} + \frac{\left(\frac{L_1}{8}(8L_2-3)+\frac{11}{16}\right)d}{(8k_2-k_1)}\right]^2 - \frac{\left(\frac{L_1}{8}(8L_2-3)+\frac{11}{16}\right)^2 d^2c^4}{8k_2-k_1}\right\} \tag{100}$$

As before, we can immediately determine the effect of the wall on the stability of the $\delta_{2,-2}$ mode for the two limiting cases, $\rho \to \infty$ and $\rho \to 1$.

From (100), in view of (89),

$$\lim_{\rho\to\infty} \delta W^{(1)} = 2\pi H\zeta^2\varepsilon^2(-.246c^4d^2) \tag{101}$$

On the other hand, from (100), in view of (91), for $\rho \simeq 1$,

$$\delta W^{(1)} = 2\pi H\zeta^2\varepsilon^2\left[\left[-\frac{4}{3}k_1^3 + O(k_1^2)\right]c^4d^2\right) \tag{102}$$

Also, from (100), the leading order part of the eigenvector associated with $\delta_{2,-2}$ is

$$\underset{\sim}{\xi}^{(1)} = 2\delta_{2,-2}r \exp\left[2i(\theta + z)\right](i\hat{r} - \hat{\theta})$$

where,

$$\delta_{2,-2} = -\frac{\left(\frac{L_1}{8}(8L_2-3) + \frac{11}{16}\right)d}{8k_2 - k_1}$$

$\underline{\delta W}$:

As we previously stated, the stability of the bifurcated equilibrium is primarily determined by the effect of $\underset{\sim}{\xi}^{(o)}$ and the eigenstate associated with $\delta_{2,-2}$ on δW.

Thus,

$$\delta W \simeq \delta W^{(o)} + \delta W^{(1)}$$

where $\delta W^{(o)}$ and $\delta W^{(1)}$ are given by (88) and (100) respectively.

It follows from (91) and (101) that the system is unstable for $\rho \to \infty$, whereas in view of (92) and (102), the

system is stable for ρ less than some critical value, ρ_c, where $1 < \rho_c < \infty$. Table 1 shows that $\delta W \geq 0$ for $\rho \leq 1.15$. Thus $\rho_c = 1.15$.

ρ	$\delta W^{(o)}$	$\delta W^{(1)}$	δW
1.1	180	-142	38
1.12	91.2	- 80.4	10.8
1.13	71.8	- 63.6	8.2
1.14	54.2	- 50.3	3.9
1.15	41.6	- 40.8	0.8
1.16	32.3	- 33.7	- 1.4
1.2	11.8	- 17.2	- 5.4

TABLE 1

Appendix

We now verify that, compared to the unstable eigenmode associated with $\delta_{2,-2}$ to leading order, all other unstable states yield instabilities of negligible magnitudes. Furthermore, in contrast to the $\delta_{2,-2}$ mode, the instabilities associated with these eigenmodes are lessened as the conducting wall is brought closer to the plasma surface.

We proceed to compute $\delta W^{(1)}$ and $\underset{\sim}{\xi}^{(1)}$ associated with the coefficients $\delta_{2,0}$, $\delta_{-2,0}$, $\delta_{4,0}$, $\delta_{-4,0}$, $\delta_{-2,-2}$, $\delta_{6,-2}$.

$\underline{\delta W^{(1)p}}$:

Again, since $\underset{\sim}{\xi}^{(1)}$, $\underset{\sim}{\xi}^{(o)}$ and $\Omega^{(1)}$, $\Omega^{(o)}$ have the same form, $\delta W_{11}^{(1)p}$ is again identical with the prebifurcated value of δW^{p} given by (50) as

$$\delta W_{11}^{(1)p} = 2\pi H \zeta^2 \varepsilon^2 \sum_{n,\ell} \{ n(\ell + n)^2 \delta^2 \} \tag{103}$$

In Table 2, we list the leading order terms of δW_{01}. Any term smaller than leading order is considered insignificant for this analysis and is designated by a cross. We use the symbol $\underset{\sim}{\xi}_n^{(o)}$ to refer to the component of the bifurcating state with n-th azimuthal wave number, i.e. $n = \pm 1, \pm 3$.

COMPUTATION OF $\delta W^{(1)p}$

		$r(\bar{n}\cdot\bar{\xi}^{(o)}_{+1})$	$r(\bar{n}\cdot\bar{\xi}^{(o)}_{+3})$
$\xi^{(1)}_{4,2}$	$\Omega^{(1)}_o$	$24cd$	X
	$\Omega^{(1)}_1$	$-24cd$	X
	$\Omega^{(1)}_2$	X	X
$\xi^{(1)}_{2,0}$	$\Omega^{(1)}_o$	$-4cd$	X
	$\Omega^{(1)}_1$	$4cd$	X
	$\Omega^{(1)}_2$	X	X
$\xi^{(1)}_{-2,0}$	$\Omega^{(1)}_o$	$-4cd$	X
	$\Omega^{(1)}_1$	$4cd$	X
	$\Omega^{(1)}_2$	X	X
$\xi^{(1)}_{+4,0}$	$\Omega^{(1)}_o$	$(\frac{111}{2} - 192\lambda)c^2d$	X
	$\Omega^{(1)}_1$	$-48c^2d$	X
	$\Omega^{(1)}_2$	$192\lambda c^2d$	X
$\xi^{(1)}_{-4,0}$	$\Omega^{(1)}_o$	$-48c^2d + c^2d/2$	$192\lambda c^2d$
	$\Omega^{(1)}_1$	$-48c^2d$	X
	$\Omega^{(1)}_2$	$-192\lambda c^2d$	X
$\xi^{(1)}_{6,-2}$	$\Omega^{(1)}_1$	$(\frac{505}{2} - 480\lambda)c^2d$	X
	$\Omega^{(1)}_1$	$-240c^2d$	X
	$\Omega^{(1)}_2$	$480\lambda c^2d$	X
$\xi^{(1)}_{-2,-2}$	$\Omega^{(1)}_o$	$-8c^2d - c^2d/2$	$96\lambda c^2d$
	$\Omega^{(1)}_1$	$+8c^2d$	X
	$\Omega^{(1)}_2$	$-96\lambda c^2d$	X
$\xi^{(1)}_{2,-2}$	$\Omega^{(1)}_o$	0	0
	$\Omega^{(1)}_1$	0	0
	$\Omega^{(1)}_2$	$c^2d/8 + 2c^2d$	X

TABLE 2

Compared to the leading order terms in Table 2, we find that all terms from $\delta W^{(1)p}_{10}$ are of lower order. Summarizing the results of Table 2

$$\delta W^{(1)p} = 2\pi H\zeta^2\varepsilon^2 \left[+ \frac{15}{2}\,\delta_{4,0}c^2d + \frac{c^2d}{2}\,\delta_{-4,0} + \frac{25}{2}\,c^2d\delta_{6,-2} - \frac{c^2d}{2}\,\delta_{-2,-2} \right] + \delta W^{(1)p}_{11} \tag{104}$$

$\underline{\delta W^{(1)V}}$:

We decompose $\delta W^{(1)V}$ as in (99), and note that once again $\delta W^{(1)V}_{11}$ is identical to the prebifurcated value given by (58). We also find that $\delta W^{(1)V}_{01} + \delta W^{(1)V}_{10}$ are of lower order than the leading order terms of $\delta W^{(1)p}$. Thus,

$$\delta W^{(1)V} = 2\pi H\zeta^2\varepsilon^2 \left\{ nL_n(\ell + n)^2\,\delta^2_{n,\ell} \right\} \quad . \tag{105}$$

$\underline{\delta W^{(1)\Sigma}}$:

From (80) and (94), making use of (24), we find that $\delta W^{(1)\Sigma}$ results in only terms of lower order than $\delta W^{(1)p}$.

Thus,

$$\delta W^{(1)\Sigma} = 0 \quad \text{to order,} \quad O(c^4d^2).$$

$\underline{\delta W^{(1)}}$:

Adding (103), (104), and (105), and diagonalizing the result,

$$\delta W^{(1)} = 2\pi H\zeta^2\varepsilon^2 \left\{ 64(1+L_4)(\delta_{4,0} + \frac{.0568}{(1+L_4)}\, c^2 d)^2 - \frac{.22}{1+L_4}\, c^4 d^2 \right.$$

$$+ 64(1+L_4)(\delta_{-4,0} + \frac{(.0039)}{1+L_4}\, c^2 d)^2 - \frac{.001}{1+L_4}\, c^2 d$$

$$+ 32(1+L_2)(\delta_{-2,-2} - \frac{.078}{1+L_2}\, c^2 d)^2 - \frac{.002}{1+L_2}\, c^2 d \qquad (106)$$

$$\left. + 96(1+L_6)(\delta_{6,-2} + \frac{.0652}{1+L_6}\, c^2 d)^2 - \frac{.407}{1+L_6}\, c^2 d \right\}$$

From (106), the leading order part of $\underset{\sim}{\xi}^{(1)}$ is given by

$$\underset{\sim}{\xi}^{(1)} = 4\delta_{4,0} r^3 \exp[i4\theta](i\hat{r} - \hat{\theta}) - 4\delta_{-4,0} r^3 \exp[-i4\theta](i\hat{r} + \hat{\theta})$$

$$- 2\delta_{-2,-2} r \exp[i(-2\theta + 2z)](i\hat{r} + \hat{\theta})$$

$$+ 6\delta_{6,-2} r^5 \exp[i(6\theta + 2z)](i\hat{r} - \hat{\theta}) \qquad (107)$$

where,

$$\delta_{4,0} = -\frac{.0568}{1+L_4}\, c^2 d$$

$$\delta_{-4,0} = -\frac{.0039}{1+L_4}\, c^2 d$$

$$\delta_{-2,-2} = \frac{.078}{1+L_2}\, c^2 d$$

$$\delta_{6,-2} = -\frac{.0652}{1+L_6}\, c^2 d$$

Also from (106), we see that with the conducting wall placed at a radius approaching infinity, i.e. $L_n \simeq 1$,

$$\delta W^{(1)} = 2\pi H\zeta^2\varepsilon^2(-.3c^4 d^2) \quad .$$

Furthermore, from (107), we see that this instability is lessened as the wall is brought closer to the plasma. Referring to the results of Table 1, the instability due to all

coefficients other than $\delta_{2,-2}$ is negligible.

Having computed $\underset{\sim}{\xi}^{(1)}$, we can now complete the computation of $\underset{\sim}{\xi}^{(2)}$. As mentioned above, we must study the interaction of $\underset{\sim}{\xi}^{(1)}$ with $\underset{\sim}{B}^{(1)}$.

We find that the Euler equation yields,

$$\nabla \times \left\{ (\underset{\sim}{B}^{(o)} \cdot \nabla \underset{\sim}{Q}^{(2)}) + (\underset{\sim}{Q}^{(2)} \cdot \nabla \underset{\sim}{B}^{(o)}) + \underset{\sim}{B}^{(1)} \cdot \nabla \underset{\sim}{Q}^{(1)} + \underset{\sim}{Q}^{(1)} \cdot \nabla \underset{\sim}{B}^{(1)} \right\} = 0$$

where,

$$\underset{\sim}{Q}^{(2)} = (\underset{\sim}{B}^{(1)} \cdot \nabla) \underset{\sim}{\xi}^{(1)} - (\underset{\sim}{\xi}^{(1)} \cdot \nabla) \underset{\sim}{B}^{(1)}$$

and,

$$\underset{\sim}{Q}^{(1)} = (\underset{\sim}{B}^{(o)} \cdot \nabla) \underset{\sim}{\xi}^{(1)} - (\underset{\sim}{\xi}^{(1)} \cdot \nabla) \underset{\sim}{B}^{(o)}$$

We conclude that

$\underset{\sim}{\xi}^{(2)}$, $\underset{\sim}{\xi}^{(1)}$, and $\underset{\sim}{\xi}^{(o)}$ have the same form up to order c^4 and therefore the $\underset{\sim}{\xi}^{(2)}$ component of the eigenvector does not affect δW to leading order $O(c^4 d^2)$.

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